Localized (distributed) control

- Localizable control: Wang, Matni, You and Doyle ACC ’14
- Localized LQR control: Wang, Matni, and Doyle CDC’14
- Output feedback? Allerton?
Another extremely toy model

• Concretely illustrate important new ideas
• Minimal complexity otherwise
• Familiar, intuitive circuit dynamics
• Emphasize role of delays

• Instability mechanism is artificial
• Comparable to biological instabilities
• … but (so far) rare in tech infrastructure
• LC circuit
• Each node = grounded capacitance
• Each link = inductance
System Model

• Assuming each L and each C has unit value, the dynamics of the system are

\[ \dot{x}(t) = Ax(t) \]

\[ A = \begin{bmatrix} 0 & M \\ -M' & 0 \end{bmatrix} \]

where \( x(t) \) is states of node voltage and link current, \( M \) is the incidence matrix of the circuit graph.

(Will reorder for plotting later.)
Discrete Time System Model

• A first order (Euler) approximation is

\[ Ad = \begin{bmatrix} I & \text{step} \times M \\ -\text{step} \times M' & I \end{bmatrix} \]

• With step = 0.2, the maximum eigenvalue of Ad is 1.0768

• Artificially create a very unstable system

• Only biology is systematically this unstable, so far.
Simplified diagram (2 states per node)

Actuated and sensed

Only sensed

Only sensed

Actuated and sensed

Only sensed
Simplified diagram (2 states per node)

Actuated and sensed

Only sensed

Only sensed

Actuated and sensed

Only sensed
Simplified diagram (2 states per node)
Nominally each has delay 1.

Simplified diagram (2 states per node)

Expensive?
0. Physical
1. Actuation

Actuated and sensed
Only sensed
Expensive?
0. Physical
1. Actuation
2. Comms speed
3. Comp speed
4. Sensing
...

Sense, comm/comp, **act.**

- Actuated and sensed
- Only sensed
Controller “plane”

Data “plane”

SDN/ODP
Sense, comm/comp, \textit{act.}
Nominally each has delay 1.
Expensive:
• physical plant
• passive stability
• actuation
• low delay (comms and comp)

Cheap:
• comms bandwidth
• compute memory
• sensing

True for cells, nets, grids, brains, but not in general
System Model

• The discrete time system equation is

\[ x[k + 1] = A_dx[k] + B_u[k] + w[k] \]

• Example: 30 C, 29 L
Open loop dynamics

Simplified diagram
Open loop

$x(t)$

$t$
Threshold to $|x(t)| < 1$
Open loop

Threshold to $|x(t)| < 1$

Disturbance propagation

$v_2(t)$

$v_9(t)$

$v_{15}(t)$

$x(t)$
Open loop

Space-time cone

plot $\log |x(t)|$

$\log |x(t)|$
Open loop

Threshold to $|x(t)| < 1$

plot $\log |x(t)|$
\[
\begin{bmatrix}
x[T]
\\
\vdots
\\
x[1]
\end{bmatrix}
= \begin{bmatrix}
A^T
\\
\vdots
\\
A
\end{bmatrix} x[0]
\]
Controller Design

Critical Issues
1. Transient LQ (H2) cost: $\Sigma(x'x+u'u)$
2. Actuator Density
3. Communication (vs plant) Speed
4. Locality/Scalability (Computation)
5. Time/space horizon
Actuator Density

- **Standard** (centralized) optimal H2 control
- No delay (initially)
- Defer other issues ($\infty$ comm, comp, sense)
- Objective: min sum ($x'x+u'u$)
- Actuator density = # actuators / # states
- Trade-off: actuator density vs norm
- Example: 30 C, 29 L
Norm - Actuator Density (normalized)

Artificially unstable system

$\text{Opt H}_2\ \text{norm}$

Actuation

sparse

dense

$\frac{1}{3}$
Actuated and sensed

Only sensed

Standard control (circa 1970)

\(\text{Comm speed} = 0 \text{ delay}\)

\(\infty\)
Opt undelay central state

Opt undelay central ctrl

Optimal Controller
+ Norm H2 optimal
− Communication undelayed
− Design/model global/huge P
− Implementation local/huge P

Sparse actuation
Color code?
Expensive?
0. Physical
1. Actuation
2. Comms speed
3. Comp speed
4. Sensing
...

Actuated and sensed

Only sensed
Nominally delay 1.

Expensive?
0. Physical
1. Actuation
2. Comms speed
3. Comp speed
4. Sensing
...

Communication speed

Versus plant speed

\[
Ad = \begin{bmatrix}
I & \text{step} \times M \\
-\text{step} \times M' & I
\end{bmatrix}
\]
Communication speed = \( \infty \)
undelay central ctrl

undelay central state

Communication speed = ∞
Communication Speed

- Distributed
- Localized
- Undelayed central

Norm

- slow
- fast

→∞
undelay central ctrl
delay distr ctrl
delay distr state
central state
**Distributed (QI) Controller**

+ Norm (H2) “small”
+ Optimal for constraints
+ Communication delayed
  - Design/model global/huge P
  - Implementation local/huge P

---

**Delay**  
**Distr**  
**State**

**Delay**  
**Distr**  
**Ctrl**
delay local ctrl

delay local state

delay distr ctrl

delay distr state
delay local

delay distr ctrl

delay distr state
Localized Controller
+ Norm (H2) “small”
+ Optimal for constraints
+ Communication delayed
+ Design/model local/small
+ Implementation local/small
+ State local

Everything is scalable.
Conjecture:
Norm bad before method breaks

Tradeoffs

Centralized

Actuator Density

Communication Speed

Norm vs. Actuator Density

Distributed

Localized
Linear equations

\[
\begin{bmatrix}
  u[T - 1] \\
  \vdots \\
  u[0]
\end{bmatrix}
\]

\[
x[T] = 0
\]

\[
x \in X
\]

\[
u \in U
\]

\[
\begin{bmatrix}
x[T] \\
\vdots \\
x[1]
\end{bmatrix} =
\begin{bmatrix}
A^T \\
\vdots \\
A
\end{bmatrix}
\begin{bmatrix}
x[0]
\end{bmatrix} +
\begin{bmatrix}
B & \cdots & A^T B
\end{bmatrix}
\begin{bmatrix}
u[T - 1] \\
\vdots \\
u[0]
\end{bmatrix}
\]
\[
\begin{bmatrix}
  x[T] \\
  \vdots \\
  x[1]
\end{bmatrix} =
\begin{bmatrix}
  A^T \\
  \vdots \\
  A
\end{bmatrix} x[0] +
\begin{bmatrix}
  B & \cdots & A^{T-1}B \\
  0 & \ddots & \vdots \\
  0 & \cdots & B
\end{bmatrix}
\begin{bmatrix}
  u[T-1] \\
  \vdots \\
  u[0]
\end{bmatrix}
\]
finite impulse response (FIR)
Local space-time controllability

\[
\begin{bmatrix}
  x[T] \\
  \vdots \\
  x[1]
\end{bmatrix}
= \begin{bmatrix}
  A^T \\
  \vdots \\
  A
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  \vdots \\
  0
\end{bmatrix}
+ \begin{bmatrix}
  B & \cdots & A^{T-1}B \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & B
\end{bmatrix}
\begin{bmatrix}
  u[T-1] \\
  \vdots \\
  u[0]
\end{bmatrix}
\]
Past delayed state needed to compute control
Past

Space-time state cone

Local space-time controllability

This can linearly constrain any optimization

\[
\begin{bmatrix}
  x[T] \\
  \vdots \\
  x[1]
\end{bmatrix}
= \begin{bmatrix}
  A^T \\
  \vdots \\
  A
\end{bmatrix} \begin{bmatrix}
  x[0] \\
  \vdots \\
  0
\end{bmatrix}
+ \begin{bmatrix}
  B & \cdots & A^{T-1}B \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & B
\end{bmatrix}
\begin{bmatrix}
  u[T - 1] \\
  \vdots \\
  u[0]
\end{bmatrix}
\]
Optimal undelayed centralized state (old)

Optimal delayed distributed (newish) (but not scalable)

Optimal delayed *localized* (very new, scalable)
AWGN in C2, L26, C29

undelay central ctrl

undelay central state

delay local ctrl

delay local state
Control

Localized Controller
+ Norm (H2) small
+ Optimal for constraints
+ Communication is delayed
+ Design/model local/small
+ Implementation local/small
+ State local

Local space-time controllability

This can linearly constrain any optimization

\[
\begin{bmatrix}
  x[1] \\
  \vdots \\
  x[T]
\end{bmatrix} =
\begin{bmatrix}
  A \\
  \vdots \\
  A^T
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  \vdots \\
  x[0]
\end{bmatrix} +
\begin{bmatrix}
  B \\
  \vdots \\
  0
\end{bmatrix}
\]
Localized Controller
+ Norm (H2) small
+ Optimal for constraints
+ Design/model is local
+ Implementation is local
+ State stays local

- Bandwidth is $\infty$

? Output feedback?
? Approximately local?
? Layering?
? Nonlinear, MPC, etc?
? Comms codesign?

See also Javad’s new relaxations
Extensions

• Scalable optimal control
  – Localizable control: Y.-S. Wang, N. Matni, S. You and J. C. Doyle ACC ’14
  – Localized LQR control: Y.-S. Wang, N. Matni, and J. C. Doyle CDC’14
  – Output feedback progress

• Dealing with varying-delays (jitter)
  – Two player LQR with varying delays: N. Matni and J. C. Doyle CDC’ 13, N. Matni, A. Lamperski and J C. Doyle IFAC ‘14
More Nikolai Matni

• Next
• Stay tuned
More Extensions/Apps

- Apps: neuro, smartgrid, CPS, cells
- IMC/RHC, etc (all of centralized control theory)
- Cyber theory: Delay jitter (uncertainty)
- Cyber: Comms co-design (CDC student prize paper)
- Physical: Robustness (unmodeled dynamics, noise)
- Cyber-phys: System ID, ML, adaptive
- SDN (Software defined nets, OpenDaylight)

- Revisit “layering as optimization”? 
- Poset causality (streamlining)?
- Quantization and network coding?
Revisit layering as optimization decomposition
Chiang, Low, Calderbank, Doyle, 2007
SDN/ODP
Layered Architectures

Controller “plane”

Cyber

Data “plane”

Physical
**Conjecture:**
Norm bad before method breaks

![Graph showing actuator density and communication speed tradeoffs](image)

- Centralized
- Distributed
- Localized
- Undelayed central

**Tradeoffs**
Communication Speed

- Delayed Centralized
- Decentralized
- Localized

Error theory

Local control theory

Speed ($=1/\infty$ delay)
Communication Speed

- Delayed Centralized
- Decentralized
- Localized

Local control theory

Error

Speed (\(=1/\text{delay}\))

\[ \text{norm} \]

\[ \infty \]

\[ \infty \]
Due to quantization, loss, noise
Local control theory

Speed (=1/delay)

Error

Rate (BW)

Communications
Local control theory

Speed (\(=1/\text{delay}\))

Rate (BW)

Error

Communications
Speed \to \infty \quad Rate \to \infty

Local control theory

\infty \quad \infty

Error \to 0 \quad \infty \quad \infty

Error \to 0

Communications
Local control
Tradeoffs

Resource 1

Resource 2

Error

Error

Ideal
Control over limited channels (Martins et al)

\[
\int f(\omega) d\omega \triangleq \frac{1}{\pi} \int_0^\infty f(\omega) d\omega
\]

b) \( P(p) = \infty \quad p \geq 0 \)

c) \( S(j\omega) \triangleq \frac{E(j\omega)}{D(j\omega)} \)

d) \( \int \log |S| d\omega \geq p - C_S \)

e) \( \int \min\left(0, \log |S|\right) d\omega \geq p - C_A \)

f) \( P(z) = 0 \Rightarrow \int \ln |S(j\omega)| \frac{z}{z^2 + \omega^2} d\omega \geq \frac{1}{2} \ln \left|\frac{z + p}{z - p}\right| \left( \geq \frac{p}{z} \text{ if } p < z \right) \)
Control over limited channels (Martins et al)

a) $e = d - u$

\[
\begin{align*}
\text{Plant} & \quad (P) \\
\text{decode/} & \quad \text{control/} \\
\text{encode} & \quad d
\end{align*}
\]

f) $P(z) = 0 \Rightarrow$

\[
\int \ln |S(j\omega)| \frac{z}{z^2 + \omega^2} d\omega \geq \frac{1}{2} \ln \left| \frac{z + p}{z - p} \right| \quad \left( \geq \frac{p}{z} \text{ if } p < z \right)
\]

b) $P(p) = \infty \quad p \geq 0$

c) $S(j\omega) \triangleq \frac{E(j\omega)}{D(j\omega)}$
Control over limited channels (Martins et al)

\[ e = d - u \]

\[ d \rightarrow \text{delay } \delta \rightarrow \text{source/disturbance} \rightarrow \text{sense/encode} \rightarrow \text{Channels} \rightarrow C_S \]

\[ \int f(\omega) d\omega \triangleq \frac{1}{\pi} \int_{0}^{\infty} f(\omega) d\omega \]

b) \[ P(p) = \infty \quad p \geq 0 \]

c) \[ S(j\omega) \triangleq \frac{E(j\omega)}{D(j\omega)} \]

d) \[ \int \log|S| d\omega \geq p - C_S \]
Control over limited channels (Martins et al)

\[
\int f(\omega) d\omega \triangleq \frac{1}{\pi} \int_0^\infty f(\omega) d\omega
\]

b) \( P(p) = \infty \quad p \geq 0 \)

c) \( S(j\omega) \triangleq \frac{E(j\omega)}{D(j\omega)} \)

e) \[
\int \min(0, \log |S|) d\omega \geq p - C_A
\]
Control over limited channels (Martins et al)

\[ \int f(\omega) d\omega \triangleq \frac{1}{\pi} \int_0^\infty f(\omega) d\omega \]

b) \( P(p) = \infty \quad p \geq 0 \)

c) \( S(j\omega) \triangleq \frac{E(j\omega)}{D(j\omega)} \)

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e) \( \int \min(0, \log|S|) d\omega \geq p - C_A \)
Control over limited channels (Martins et al)

\[ \int f(\omega)d\omega \triangleq \frac{1}{\pi} \int_{0}^{\infty} f(\omega)d\omega \]

b) \[ P(p) = \infty \quad p \geq 0 \]

c) \[ S(j\omega) \triangleq \frac{E(j\omega)}{D(j\omega)} \]

d) \[ \int \log |S|d\omega \geq p - C_S \]

e) \[ \int \min(0, \log |S|)d\omega \geq p - C_A \]

f) \[ P(z) = 0 \Rightarrow \int \ln |S(j\omega)| \frac{z}{z^2 + \omega^2} d\omega \geq \frac{1}{2} \ln \left| \frac{z + p}{z - p} \right| \geq \frac{p}{z} \text{ if } p < z \]
Universal laws and architectures (Turing)

Architecture (constraints that deconstrain)

Architecture (laws (law))

Speed

Fast

Slow

General

Special

Ideal
Memory is cheap, reusable, powerful. Time is not.
- Cheap: memory, bandwidth, sensors
- Not: time (1/speed), actuators
- Brains/bodies, cells, CyberPhySys, …
All costs are ultimately “physical.”