Control, Dynamical Systems, and Fluids
Balanced models and the Koopman operator

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6 Aug 2014

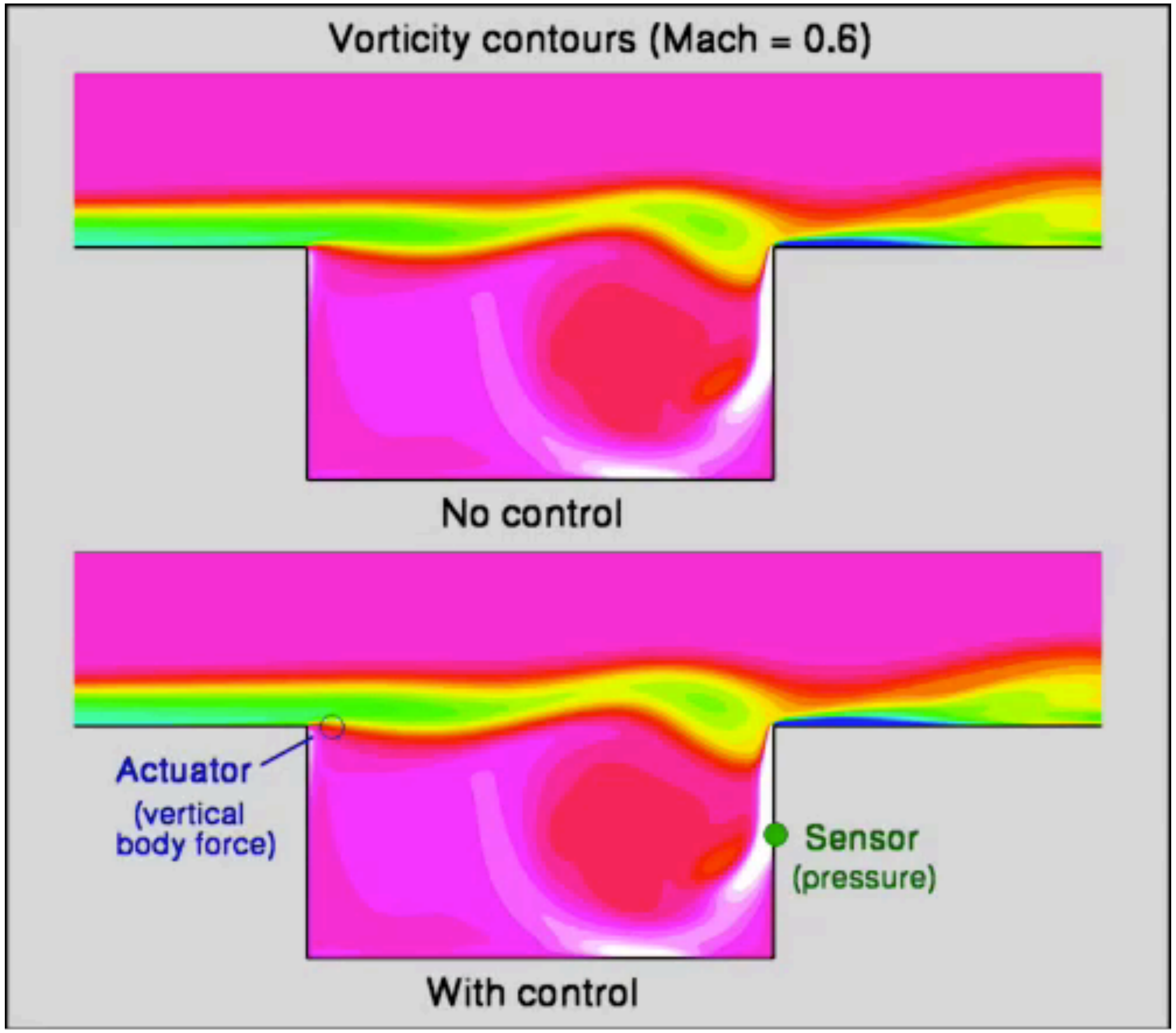
CDS20, Caltech

Mechanical
and Aerospace
Engineering
PRINCETON

Motivating example: control of cavity oscillations

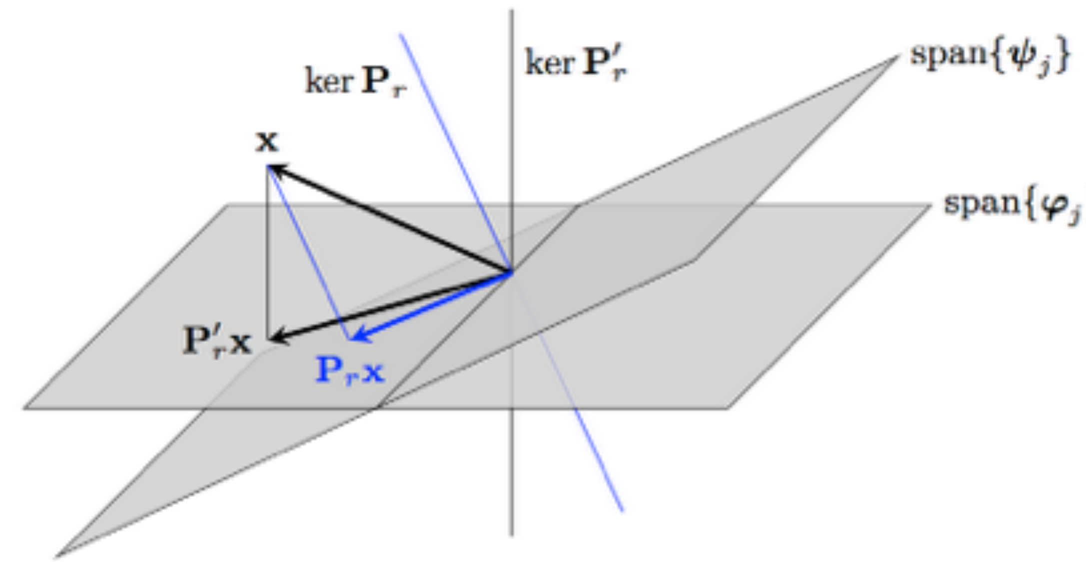
- Model-based controller to suppress oscillations
- Full system: 2,000,000 states
- Control design based on low-order model with 2 states

[Rowley and Juttijudata, CDC 2005]

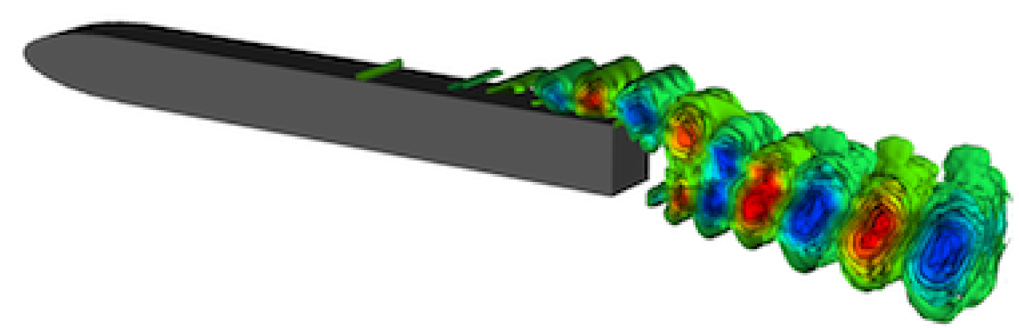


Outline

- Balanced models
  - Galerkin projection
  - Balanced truncation for fluids
  - Example: channel flow



- Koopman operator
  - Some history
  - Koopman modes
  - Examples

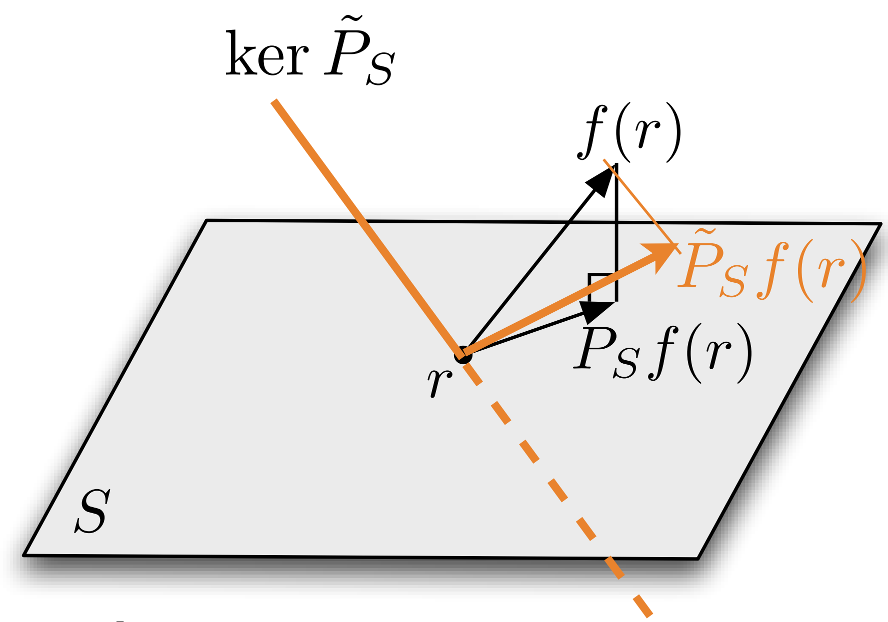


Galerkin projection

- Dynamics evolve on a high-dimensional space (or infinite-dim'l)
- Project dynamics onto a low-dimensional subspace S

$$\dot{x} = f(x) \qquad x \in V$$

$$r \in S \subset V$$



- Define dynamics on the subspace by

$$\dot{r} = P_S f(r) \qquad P_S : V \to S$$ is a projection

- Two choices:
  - choice of subspace
  - choice of inner product
    (equivalently, choice of the nullspace for a non-orthogonal projection)

Balanced truncation for fluids?

- Why not use balanced truncation?
  - Typically intractable for fluids simulations
  - Requires simultaneous diagonalization of controllability and observability Gramians (non-sparse symmetric n x n matrices). Even storing these is usually not possible:

Dimension	Grid size	n	Storage
1	100	10	39 KB
2	100 x 100	10	381 MB
3	100 x 100 x 100	10	3.7 TB

Key idea: empirical Gramians

- Approximate the Gramians using data from simulations
  [Lall, Marsden, Glavaski, IFAC 1999]
- Controllability Gramian:

$$\dot{x} = Ax + Bu$$

$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t}\, dt$$

For a single-input system, let $x(t) = e^{At}B$ be solution for $x(0) = B$

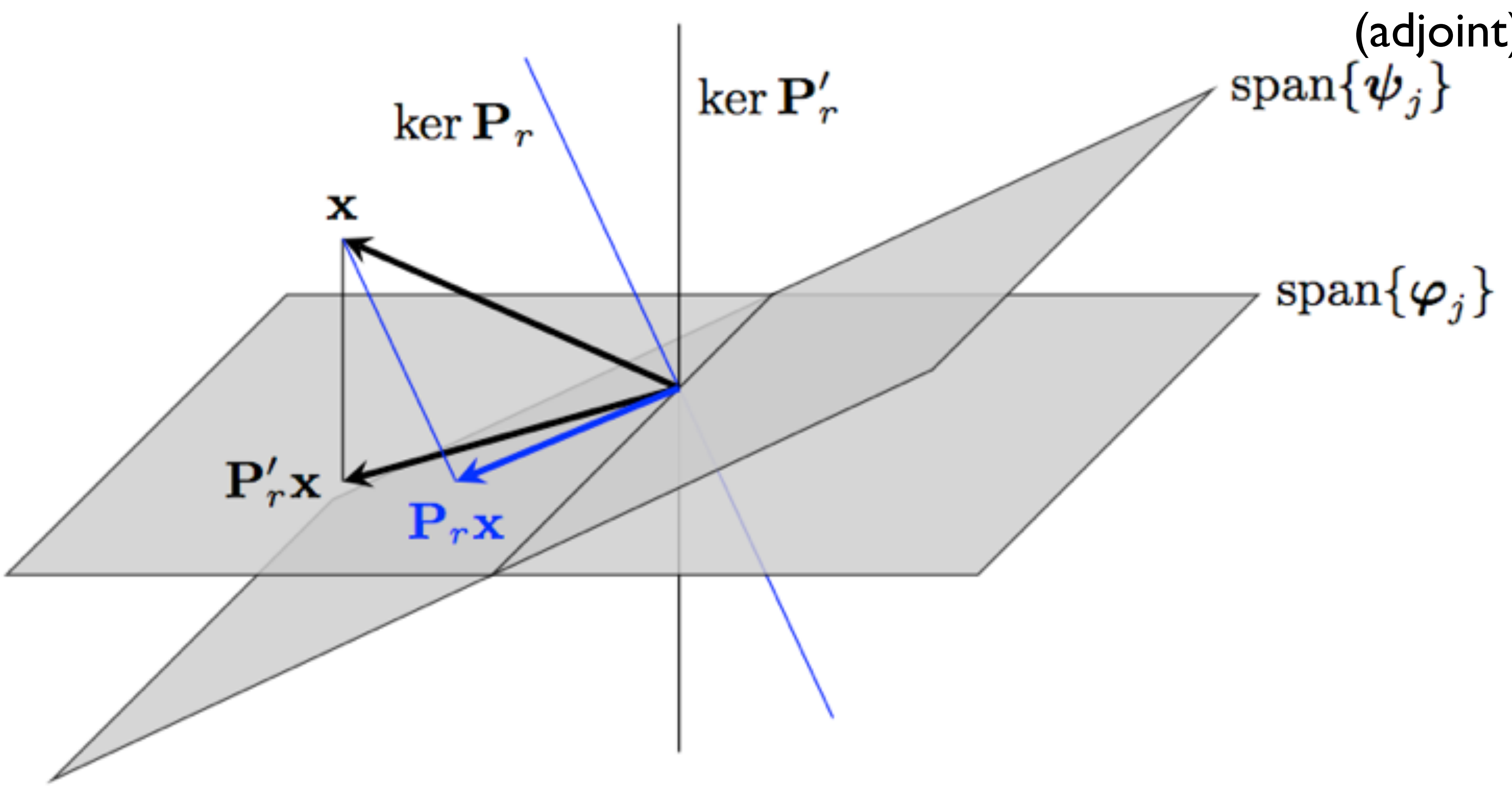
$$W_c = \int_0^\infty x(t) x(t)^T\, dt \approx \sum_{k=0}^{m} x_k x_k^T\, \Delta t$$

- Balanced POD: algorithm for computing balancing transformation from data, analogous to "method of snapshots" for POD.
  [Rowley, IJBC 2005]

Net result: can compute the balancing transformation directly from data, even for high-dimensional systems.

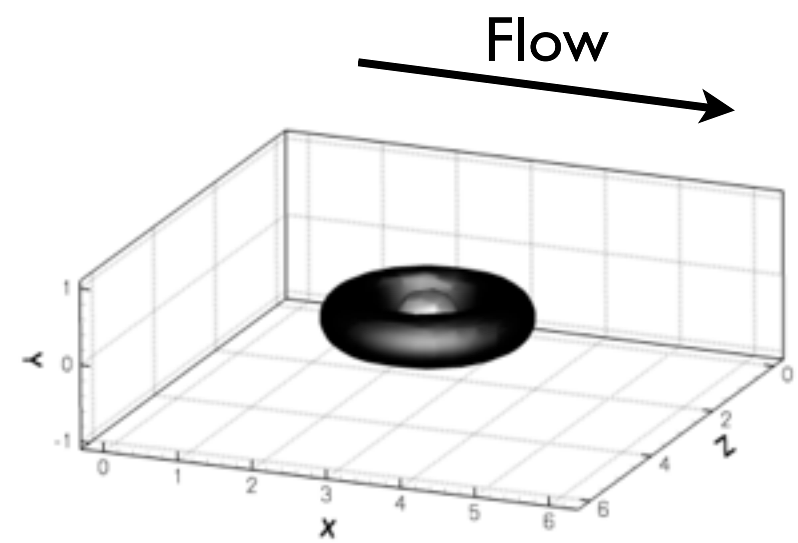
Geometry of the projection

- Balanced POD gives two sets of modes
- Balancing modes φ_j determine the subspace to project onto
  - These are typically close to the POD modes
- Adjoint modes ψ_j determine the direction of projection
  - Typically very different from POD modes



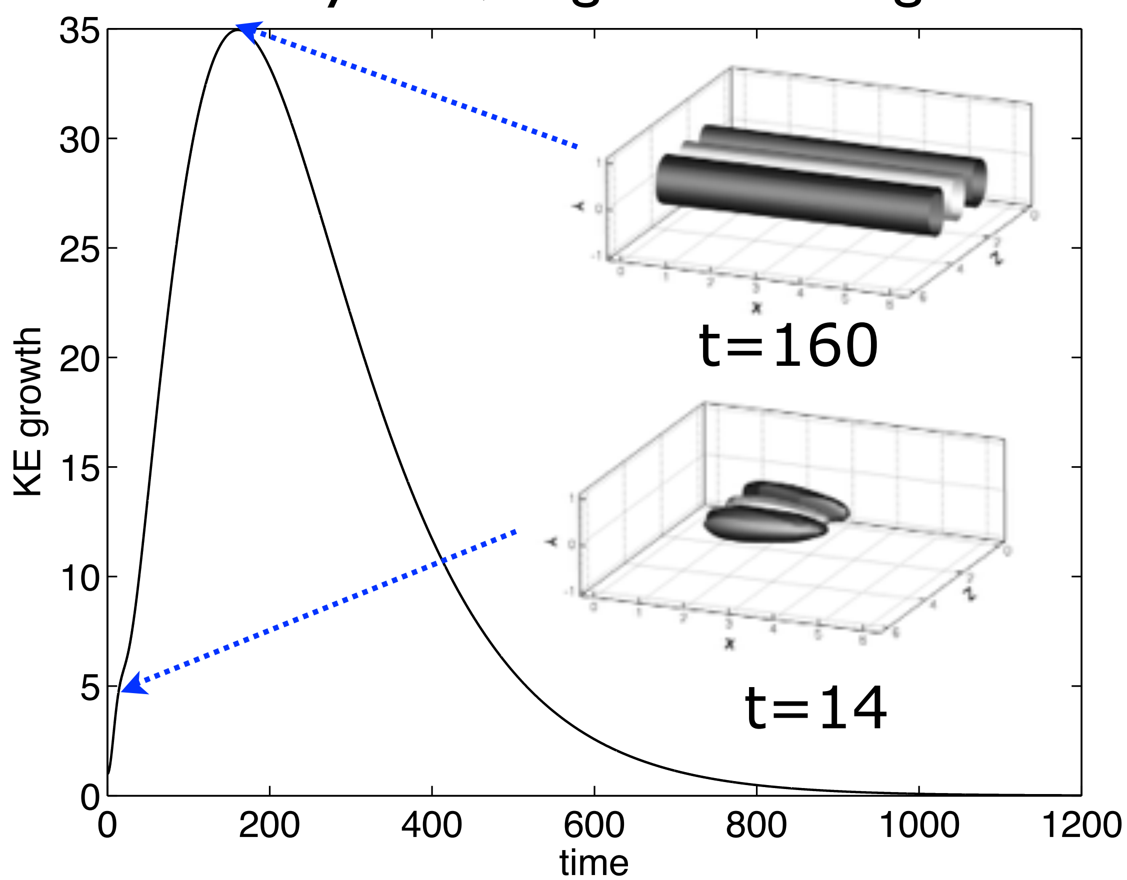
Linearized channel flow

- Plane channel flow in a periodic box
  - Consider linear development of small perturbations
  - Stable system, but large transient growth (non-normal)
- Approach
  - Compare POD with Balanced POD
  - DNS, Re = 2000, 32x65x32 grid, 133,120 states
  - Try to capture linear dynamics with a reduced-order model

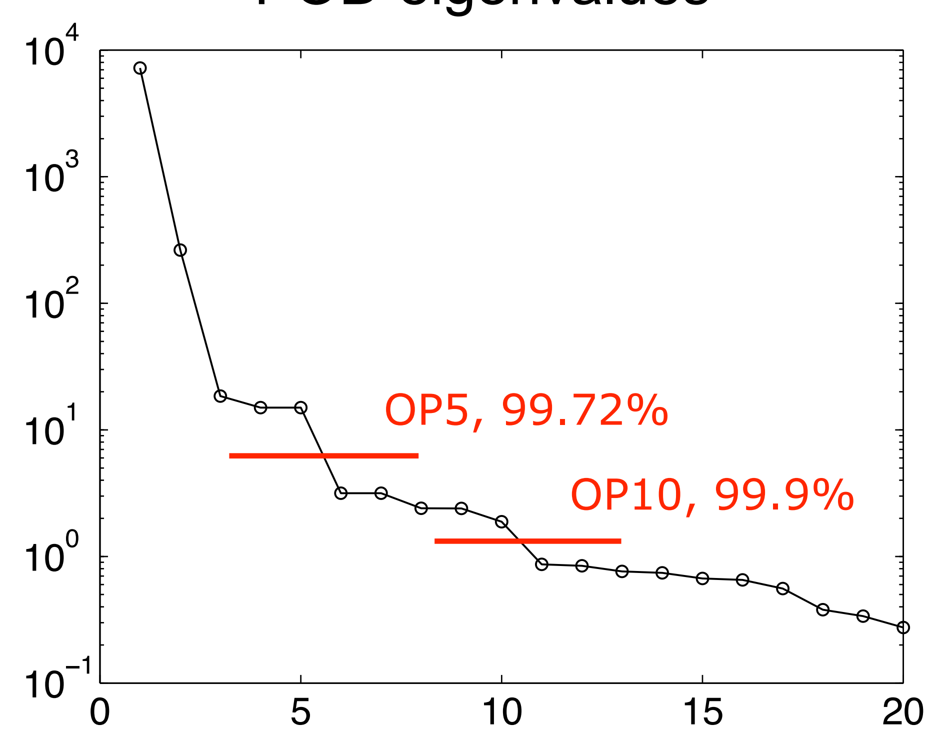


initial condition
(vertical velocity)

Stable system, large transient growth

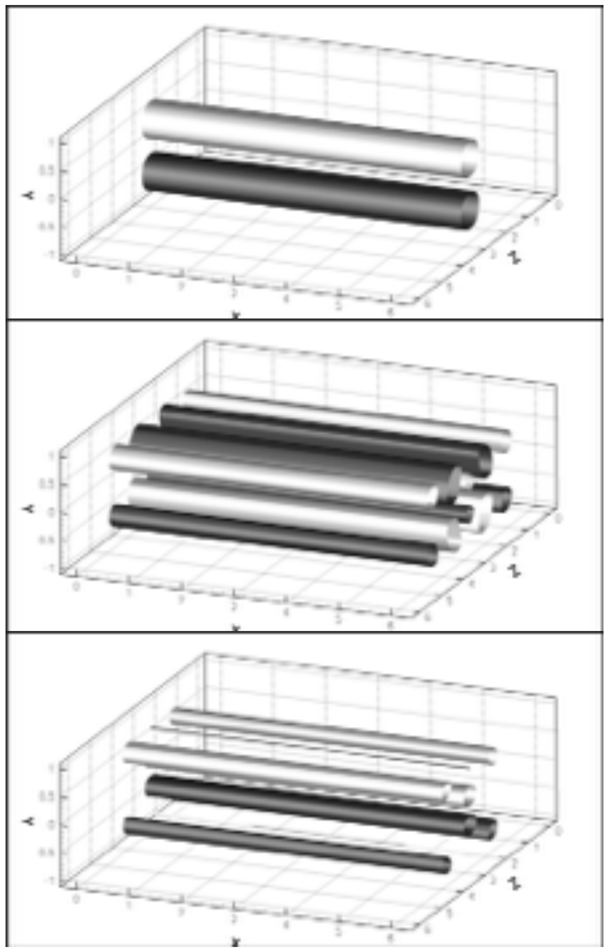


POD eigenvalues

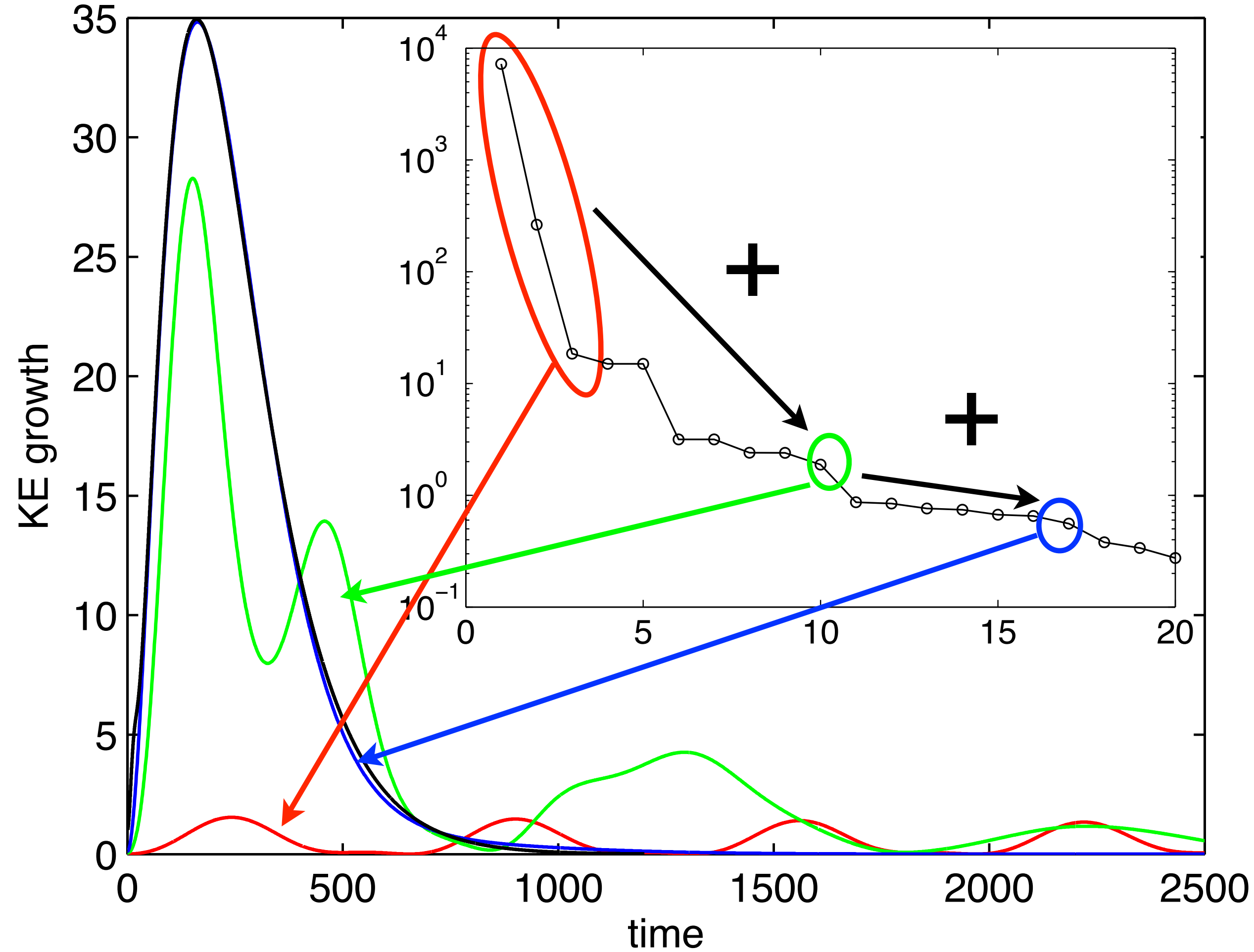


POD model performance

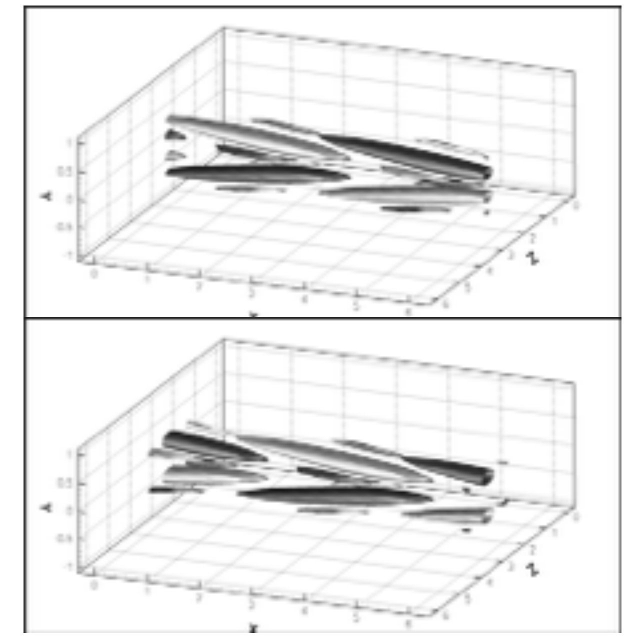
POD modes 1-3



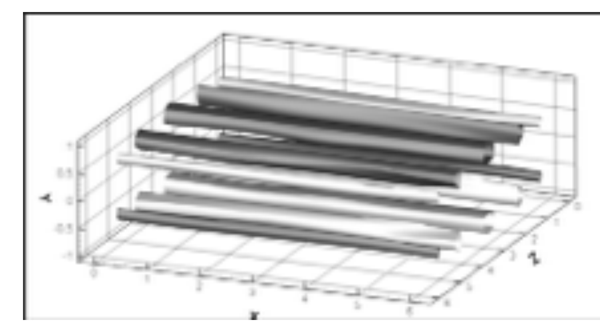
Standard POD



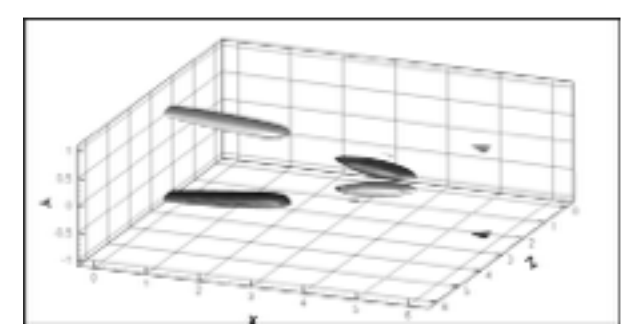
POD modes 4-5



POD mode 10



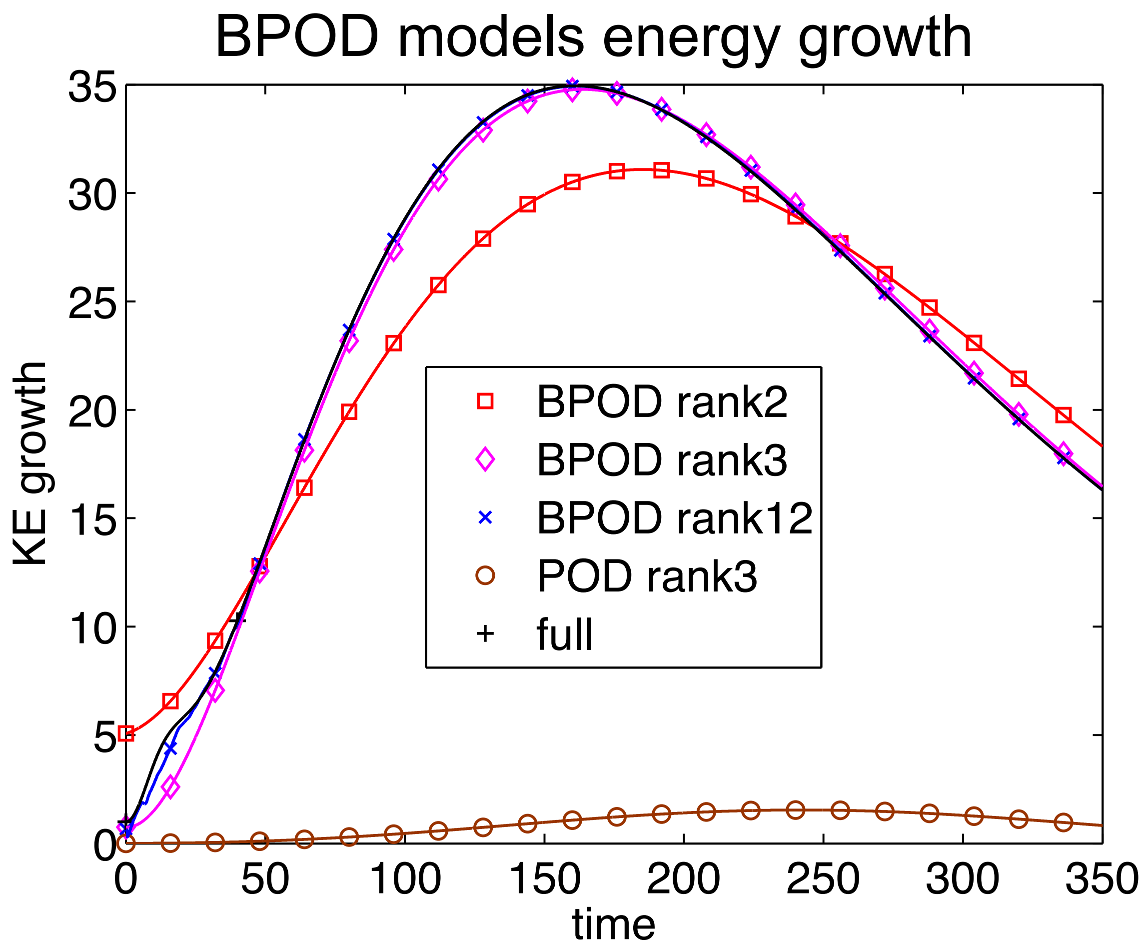
POD mode 17

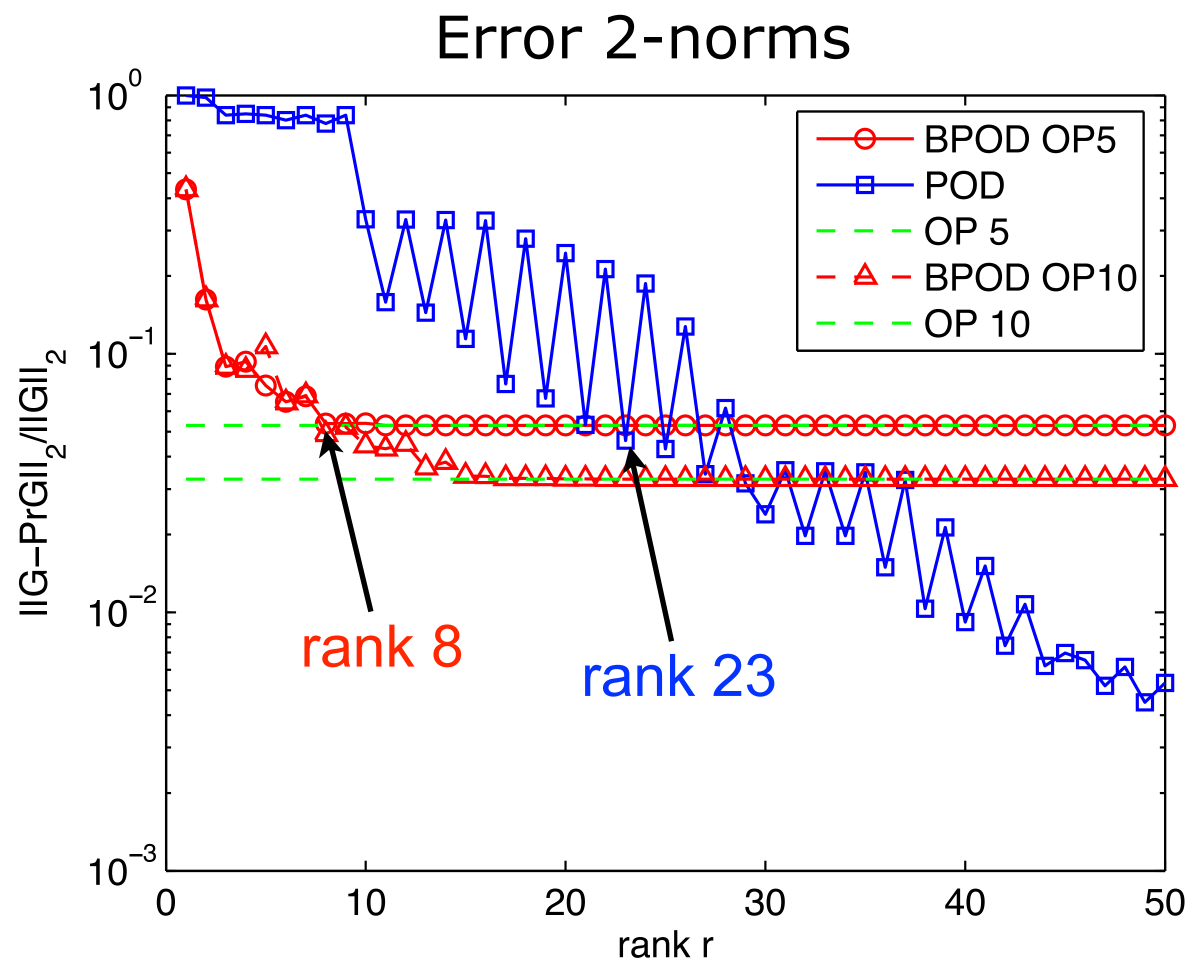


- 5-order model with modes 1,2,3,10,17 much better than 5-mode model with modes 1–5.

Conclusion: some low-energy POD modes are very important for the system dynamics. Can't naively use just the most energetic ones.

Balanced POD models are more accurate

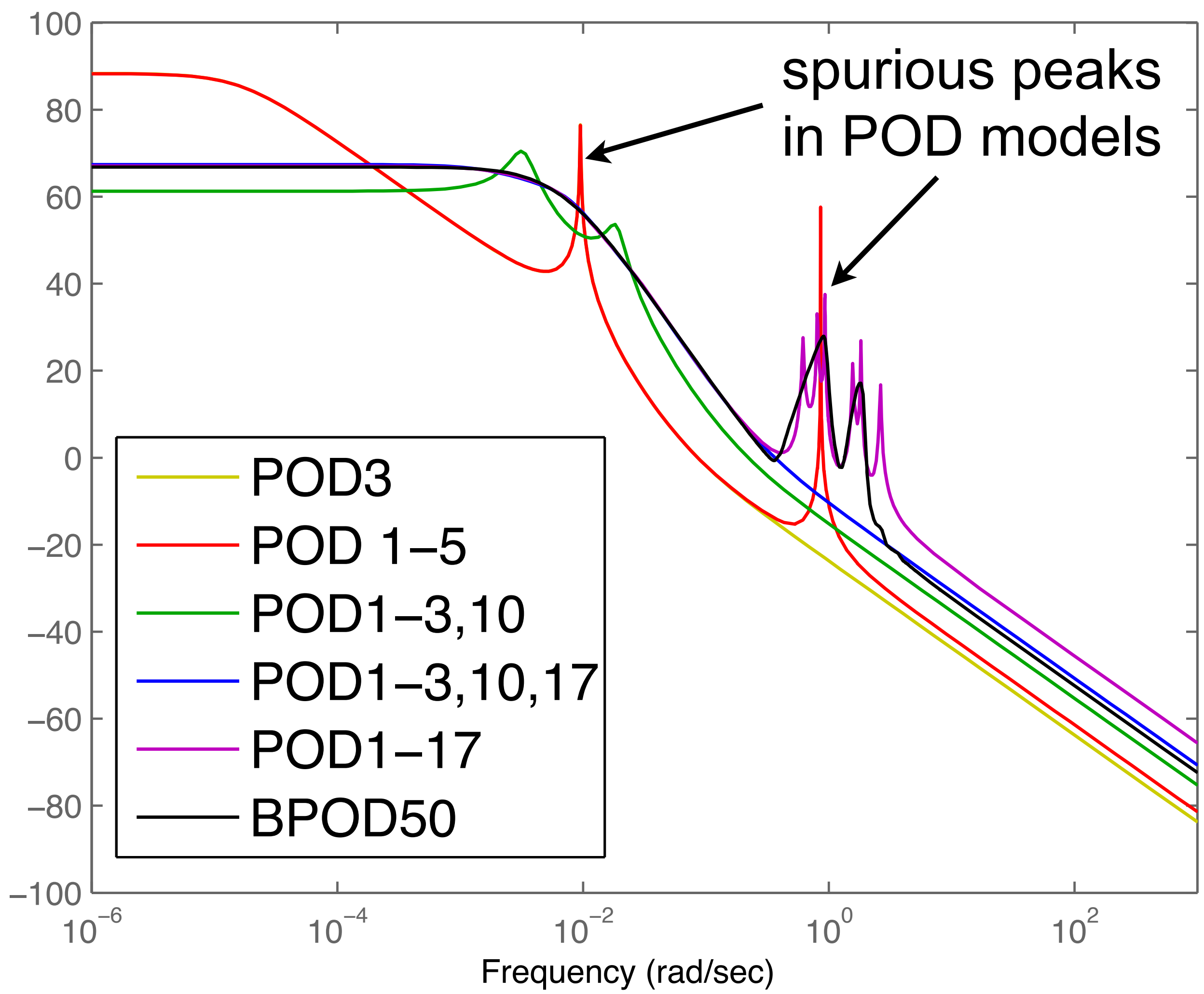




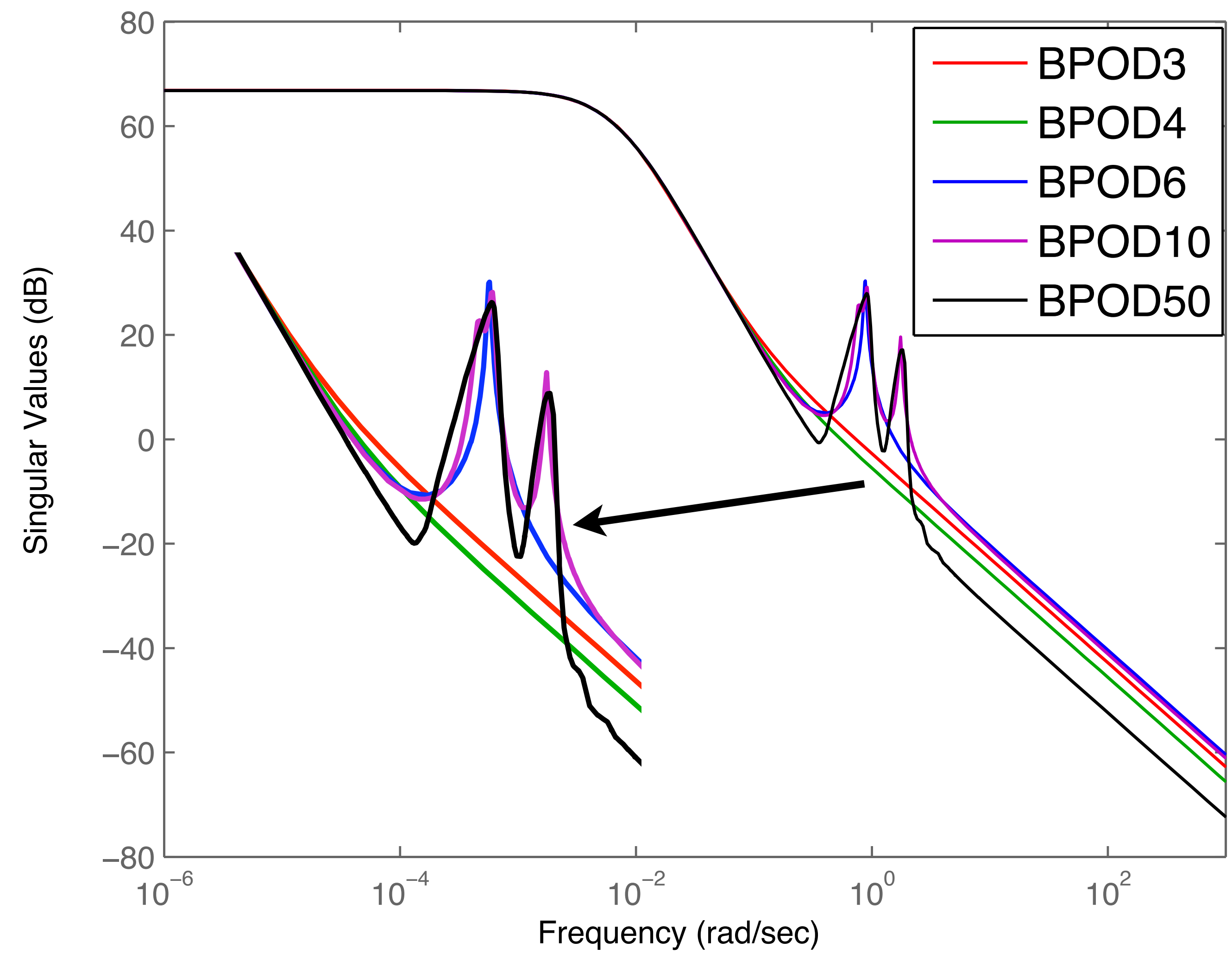
- Three-mode BPOD model excellent at capturing the energy growth
- Rank 8 BPOD model sufficient to correctly capture the dynamics of the first five POD modes, compared to at least 23 POD modes
- Explanation: BPOD weights modes by their observability, or dynamical importance, not just energy

Balanced POD models are less fragile

POD singular value Bode plot



BPOD singular value Bode plot



- BPOD 10-mode OP 50-mode model taken as 'full system'
- POD poorly captures low-pass behavior, spurious peaks
- BPOD models more "robust" than POD (no spurious lightly-damped modes)

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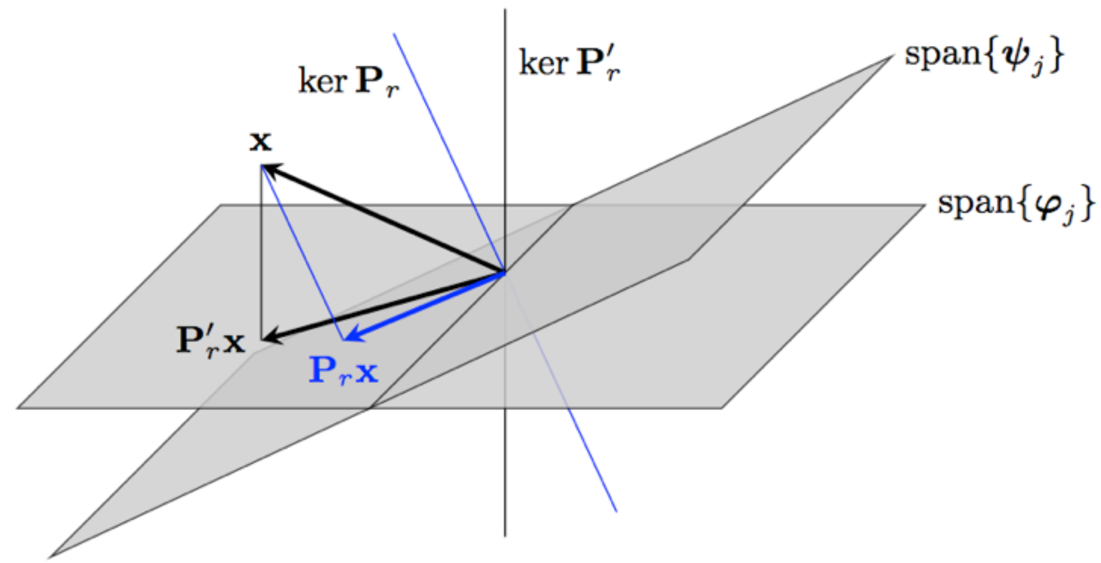
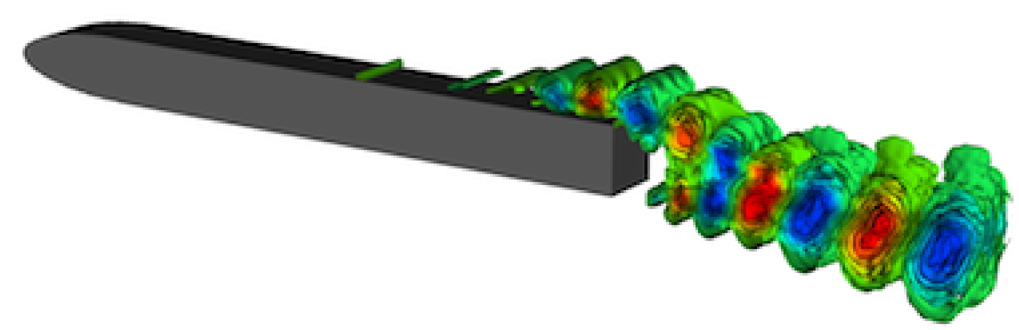


- Koopman operator
  - Some history
  - Koopman modes
  - Examples



Koopman operator

- Consider a nonlinear system (discrete time)

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

- Define the Koopman operator U as follows: for any scalar-valued function g, U maps g into a new function

$$Ug(\mathbf{x}) = g(\mathbf{f}(\mathbf{x}))$$

- U is linear:

$$U(\alpha g_1 + \beta g_2)(\mathbf{x}) = \alpha U g_1(\mathbf{x}) + \beta U g_2(\mathbf{x})$$

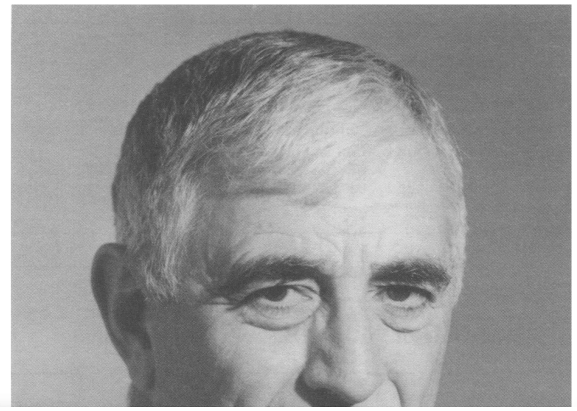
- Koopman (1931) proposed analyzing a dynamical system by studying the spectral properties of U

- Historical digression...

George David Birkhoff (1884–1944)

- Probably the preeminent American mathematician of his time
- Taught at Wisconsin 1907–9
- Moved to Princeton in 1909 as a preceptor in Mathematics; became a professor in 1911
- Moved to Harvard in 1912
- 1931: proved the ergodic theorem (age 47)

Bernard Osgood Koopman (1900–1981)



VOL. 17, 1931 MATHEMATICS: B. O. KOOPMAN 315

HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

BY B. O. KOOPMAN

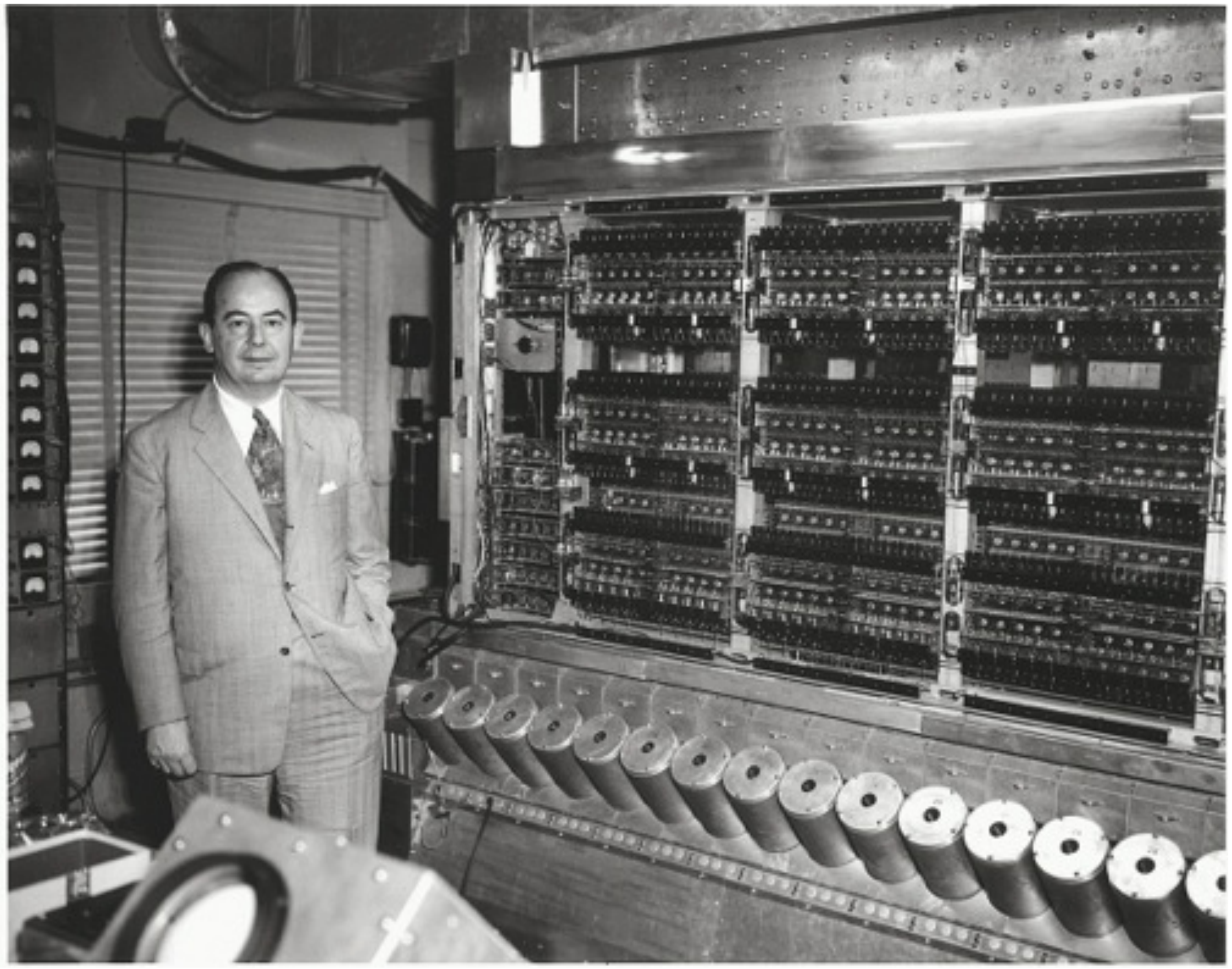
DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated March 23, 1931

In recent years the theory of Hilbert space and its linear transformations has come into prominence.[1] It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions—

John von Neumann (1903–1957)

- 1930: Moved to Princeton (from Germany) as an assistant professor in Mathematics
- 1931/2: proved the mean ergodic theorem (age 28)



John von Neumann by the IAS machine in 1952

The ergodic theorems

- Mean ergodic theorem (von Neumann, PNAS 1931/2)
  - His first paper published in English
  - Koopman translated it from German

PROOF OF THE QUASI-ERGODIC HYPOTHESIS

By J. v. Neumann

Department of Mathematics, Princeton University

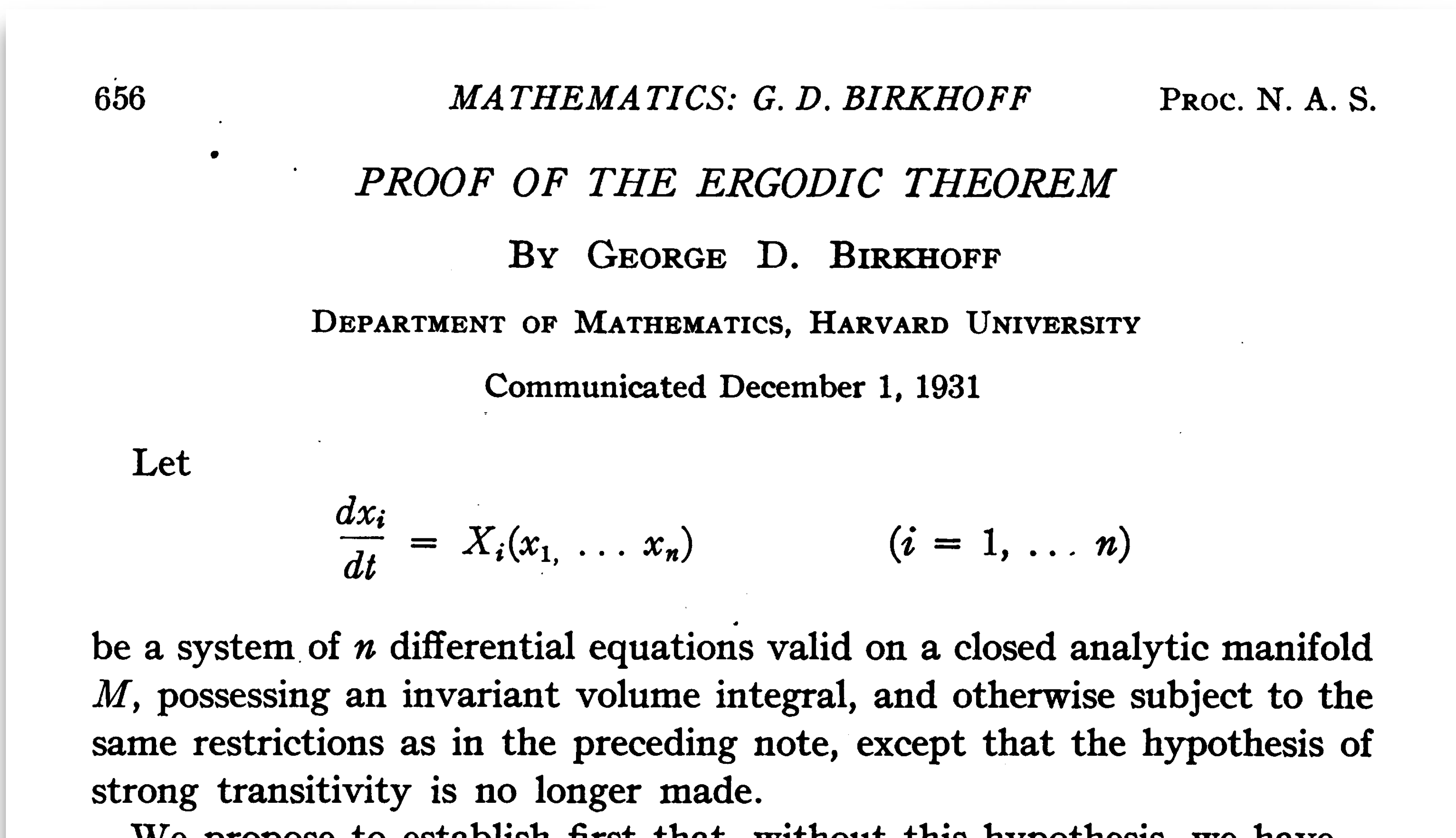
Communicated December 10, 1931

1. The purpose of this note is to prove and to generalize the quasi-ergodic hypothesis of classical Hamiltonian dynamics[1] (or "ergodic hypothesis," as we shall say for brevity) with the aid of the reduction, recently discovered by Koopman,[2] of Hamiltonian systems to Hilbert space, and with the use of certain methods of ours closely connected with recent investigations of our own of the algebra of linear transformations in this space.[3] A precise statement of our results appears on page 79.

We shall employ the notation of Koopman's paper, with which we assume the reader to be familiar. The Hamiltonian system of k degrees of freedom corresponding with the Hamiltonian function $H(q_1, \ldots, q_k,$

Birkhoff's ergodic theorem

- von Neumann obtained his theorem first, and Birkhoff was aware of his result
- Birkhoff's paper appeared in Dec 1931, von Neumann's in Jan 1932!
- Birkhoff and Koopman wrote a paper [PNAS, Mar 1932] setting the record straight.
- Detailed account: [J. Zund, Historia Mathematica 2002]



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PROOF OF THE ERGODIC THEOREM

BY GEORGE D. BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated December 1, 1931

Let

$$\frac{dx_i}{dt} = X_i(x_1, \ldots x_n) \qquad (i = 1, \ldots n)$$

be a system of n differential equations valid on a closed analytic manifold M, possessing an invariant volume integral, and otherwise subject to the same restrictions as in the preceding note, except that the hypothesis of strong transitivity is no longer made.

We propose to establish first that, without this hypothesis, we have

Meanwhile, at Bell Labs...

- Nyquist stability criterion (1932)

Regeneration Theory

By H. NYQUIST

Regeneration or feed-back is of considerable importance in many applications of vacuum tubes. The most obvious example is that of vacuum tube oscillators, where the feed-back is carried beyond the singing point. Another application is the 21-circuit test of balance, in which the current due to the unbalance between two impedances is fed back, the gain being increased until singing occurs. Still other applications are cases where portions of the output current of amplifiers are fed back to the input either unintentionally or by design. For the purpose of investigating the stability of such devices they may be looked on as amplifiers whose output is connected to the input through a transducer. This paper deals with the theory of stability of such systems.

Preliminary Discussion

WHEN the output of an amplifier is connected to th a transducer the resulting combination may be unstable. The circuit will be said to be stable when an disturbance, which itself dies out, results in a response

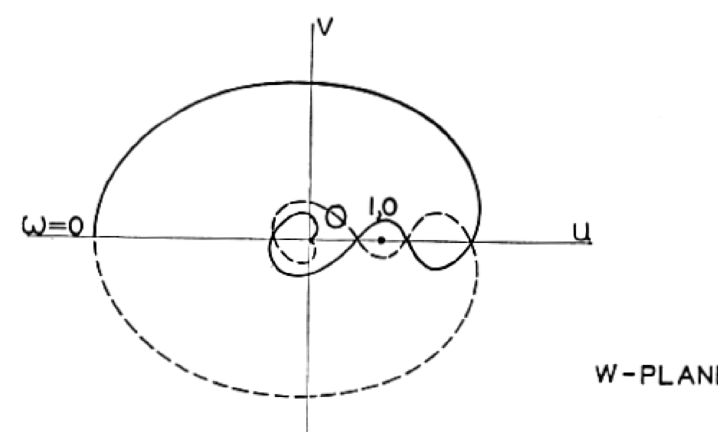


Fig. 3—Illustrating case where amplifying ratio is real and greater than unity for two frequencies, but where nevertheless the path of integration does not include the point 1, 0.

Koopman eigenfunctions and modes

- Consider a nonlinear system (discrete time)

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

and the corresponding Koopman operator U:

$$Ug(\mathbf{x}) = g(\mathbf{f}(\mathbf{x}))$$

- Suppose U has eigenvalues and eigenfunctions

$$U\varphi_j(\mathbf{x}) = \lambda_j \varphi_j(\mathbf{x}), \qquad j = 1, 2, \ldots$$

- Now, consider a vector-valued observable g(x). Expand g:

$$\mathbf{g}(\mathbf{x}) = \sum_{j=1}^{\infty} \varphi_j(\mathbf{x}) \mathbf{v}_j.$$

Koopman modes

Mezic, Nonlinear Dynamics, 2005
Rowley, Mezic, Bagheri, Schlatter, Henningson, J Fluid Mech, 2009

Example: Koopman modes for linear systems

- Consider a linear system

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

  - Find eigenvalues and eigenvectors (direct and adjoint):

$$\mathbf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j \qquad \mathbf{A}^*\mathbf{w}_j = \bar{\lambda}_j \mathbf{w}_j$$

  - Define

$$\varphi_j(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w}_j \rangle$$

  - Then φ_j are eigenfunctions of Koopman operator U:

$$\begin{aligned} U\varphi_j(\mathbf{x}) &= \varphi_j(\mathbf{A}\mathbf{x}) = \langle \mathbf{A}\mathbf{x}, \mathbf{w}_j \rangle = \langle \mathbf{x}, \mathbf{A}^*\mathbf{w}_j \rangle \\ &= \lambda_j \langle \mathbf{x}, \mathbf{w}_j \rangle = \lambda_j \varphi_j(\mathbf{x}) \end{aligned}$$

  - For the observable g(x) = x, the Koopman modes are the eigenvectors v_j:

$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{w}_j \rangle \mathbf{v}_j = \sum_{j=1}^{n} \varphi_j(\mathbf{x}) \mathbf{v}_j$$

Koopman modes

Example: Koopman modes for a periodic system

- Suppose we have a nonlinear system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with a periodic solution $\{\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}\}$

- A common way to analyze periodic solutions is the discrete Fourier transform: define a new set of vectors

$$\{\hat{\mathbf{x}}_0, \ldots, \hat{\mathbf{x}}_{m-1}\} \qquad \mathbf{x}_k = \sum_{j=0}^{m-1} e^{2\pi ijk/m}\hat{\mathbf{x}}_j$$

Koopman modes

- Define a function on the finite set $\{\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}\}$

$$\varphi_j(\mathbf{x}_k) = e^{2\pi ijk/m}$$

- These are eigenfunctions of the Koopman operator U:

$$\begin{aligned} U\varphi_j(\mathbf{x}_k) &= \varphi_j(\mathbf{f}(\mathbf{x}_k)) = \varphi_j(\mathbf{x}_{k+1}) \\ &= e^{2\pi ij(k+1)/m} = e^{2\pi ij/m}\varphi_j(\mathbf{x}_k) \end{aligned}$$

- Koopman modes are the discrete Fourier transform of the data

Computing Koopman modes: DMD

- Dynamic Mode Decomposition (DMD)
  - Introduced by Peter Schmid [APS 2008]
  - Algorithm for computing Koopman modes [Rowley et al, JFM 2009]
  - Suppose we have a discrete-time dynamical system $\mathbf{z} \mapsto \mathbf{f}(\mathbf{z})$ and two sets of data:

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_m \end{bmatrix} \qquad Y = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_m \end{bmatrix}$$

    with $\mathbf{y}_j = \mathbf{f}(\mathbf{x}_j)$

  - The DMD modes (or Koopman modes) are eigenvectors of

$$A = YX^{+}$$

    where + denotes the Moore-Penrose pseudoinverse.

  - Each DMD mode $\mathbf{v}_j$ evolves according to $\mathbf{v}_j \mapsto \lambda_j \mathbf{v}_j$

J. Tu et al, "On Dynamic Mode Decomposition: Theory and Applications," arXiv:1312:0041 [math.NA].

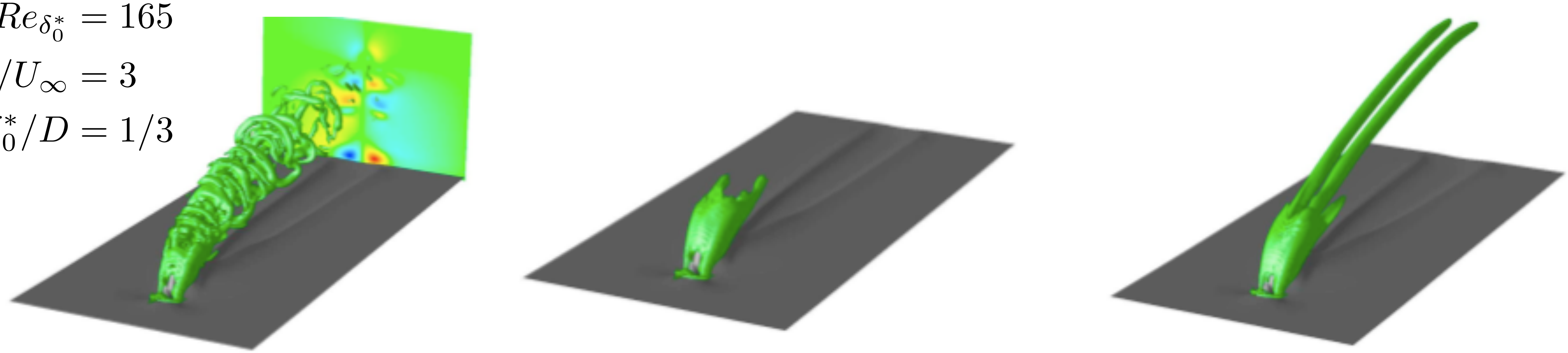
Example: jet in crossflow

- Linearize a jet in crossflow about an unstable equilibrium

$Re_{\delta_0^*} = 165$

$V_{\text{jet}}/U_\infty = 3$

$\delta_0^*/D = 1/3$



Instantaneous snapshot | Mean | Unstable equilibrium

- Compute global modes, compare frequencies with observed frequencies in shear layer and near-wall fluctuations

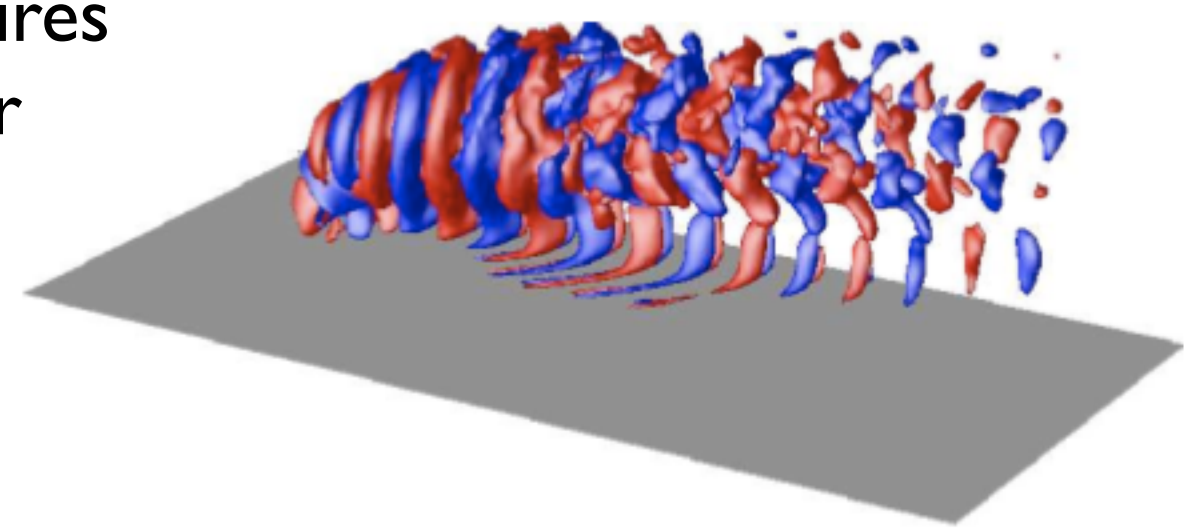
	Observed	Global mode
Shear layer	St = 0.141	St = 0.169
Near wall	St = 0.0174	St = 0.043

Frequency mismatch

[Bagheri, Schlatter, Schmid, Henningson, JFM 2009]

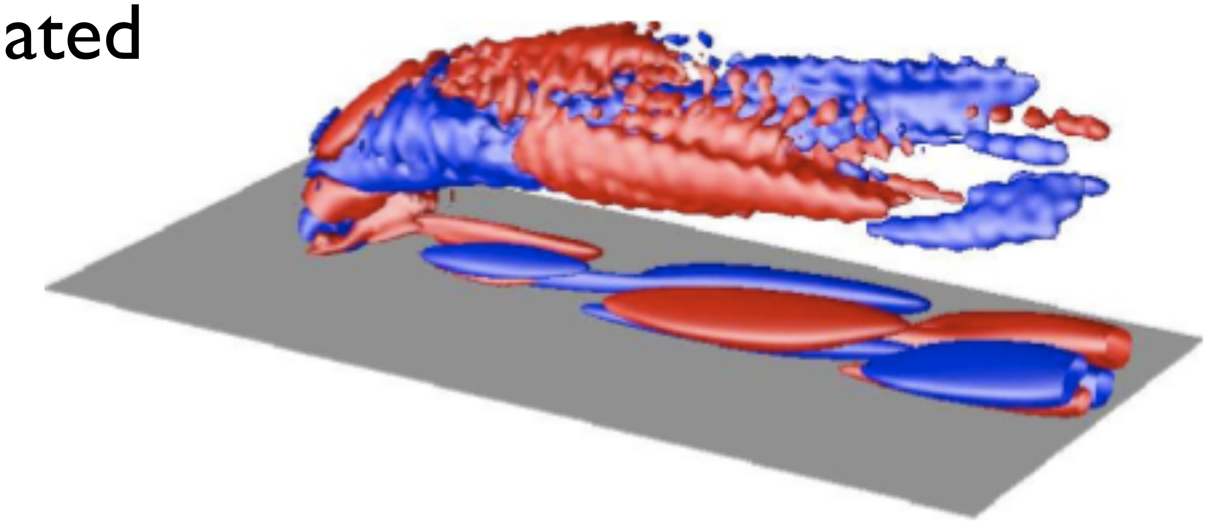
Koopman modes

- High-frequency mode captures structures in the shear layer



St = 0.141

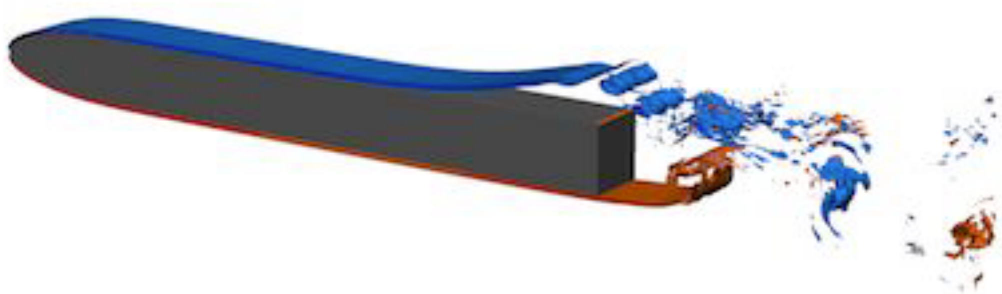
- Low-frequency mode captures near-wall structures associated with horseshoe vortex



St = 0.017

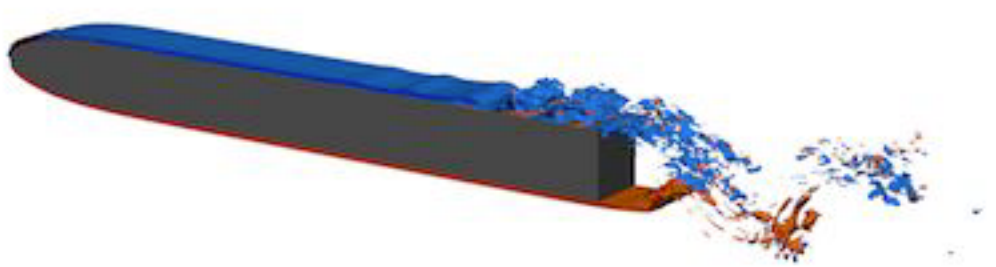
Separated flat plate

- No forcing



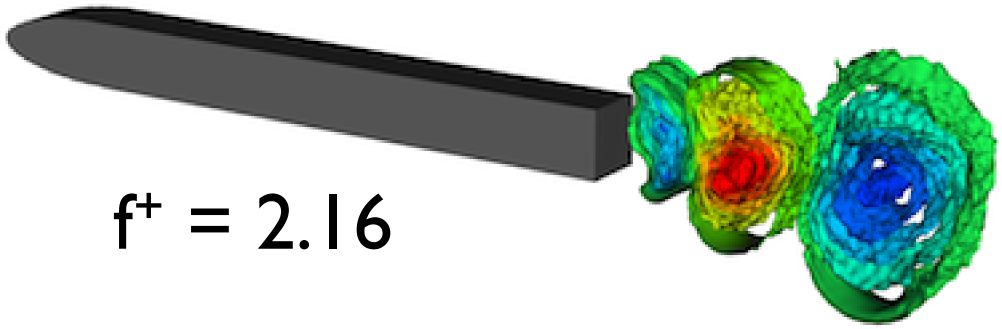
- Forcing at $f^+ = 4.7$

- Forcing at $f^+ = 6.4$

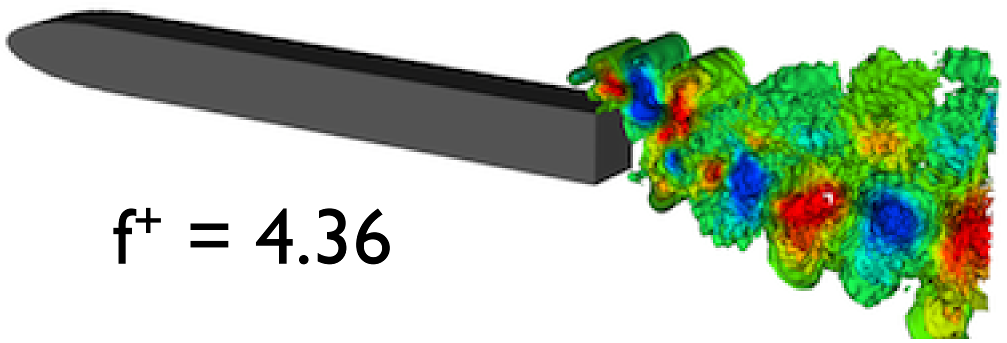


Koopman modes

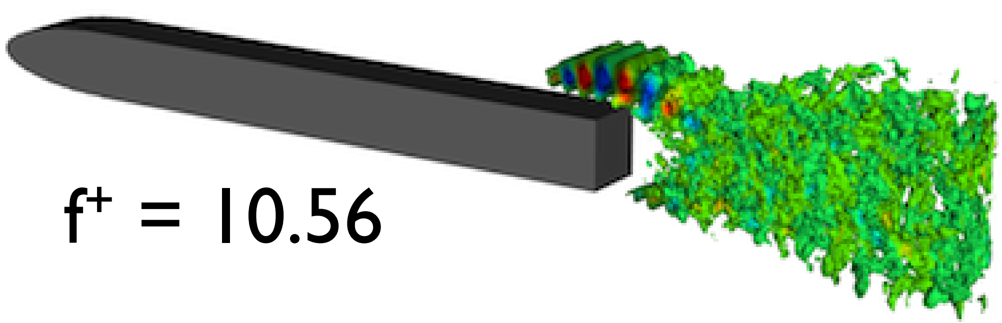
No forcing



$f^+ = 2.16$

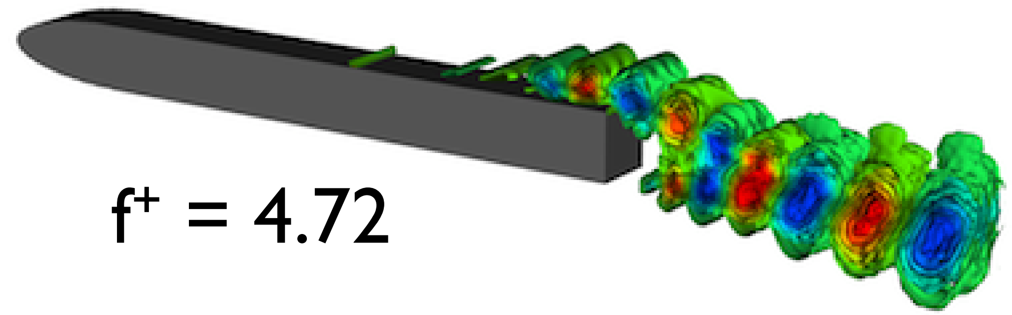


$f^+ = 4.36$

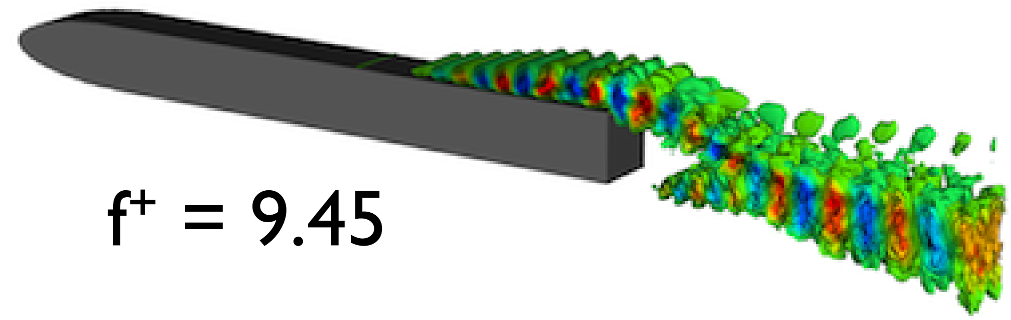


$f^+ = 10.56$

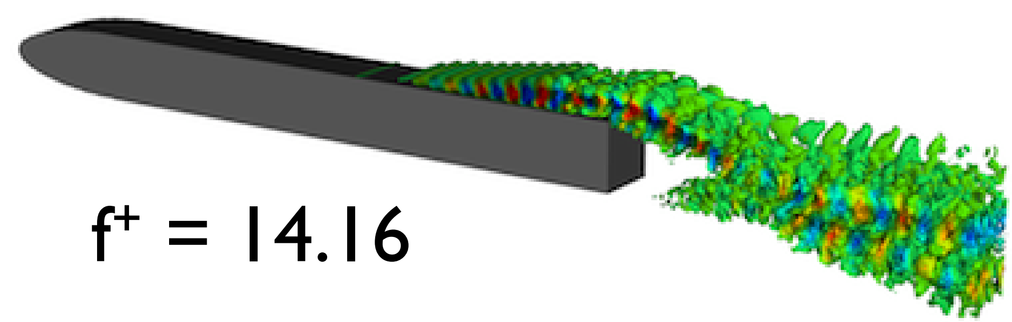
Forcing at $f^+ = 4.7$



$f^+ = 4.72$



$f^+ = 9.45$

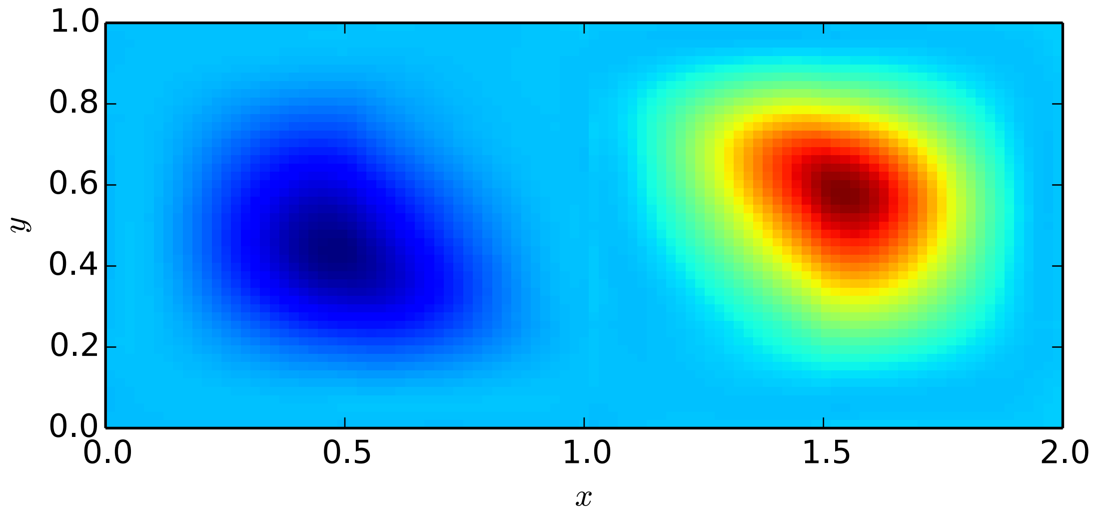


$f^+ = 14.16$

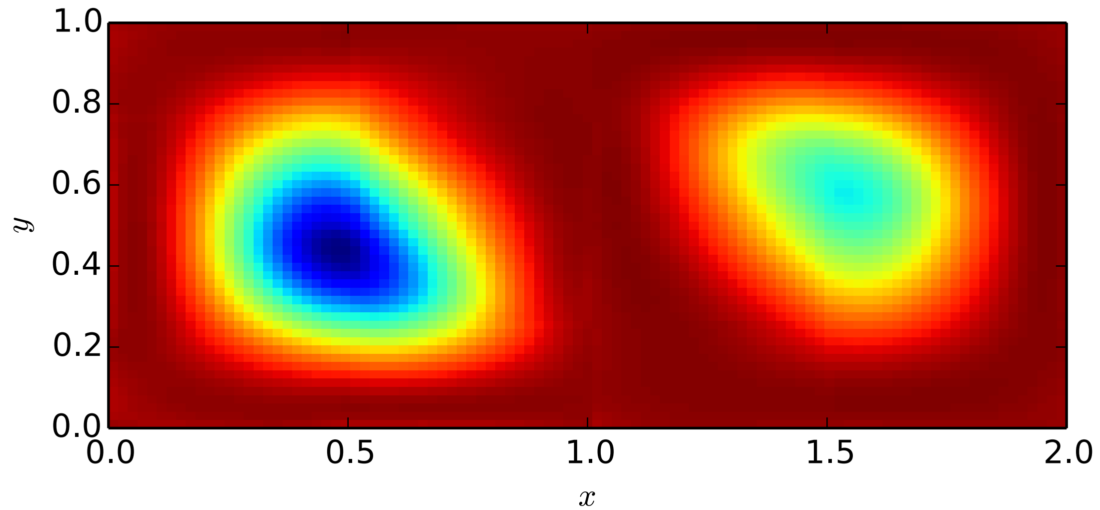
Almost invariant sets in the double gyre

- Approximate Koopman operator using (extended) DMD
- Compute almost-invariant sets, which are related to eigenfunctions of a modified Koopman operator
  [Froyland & Dellnitz, 2003] [Froyland 2005]

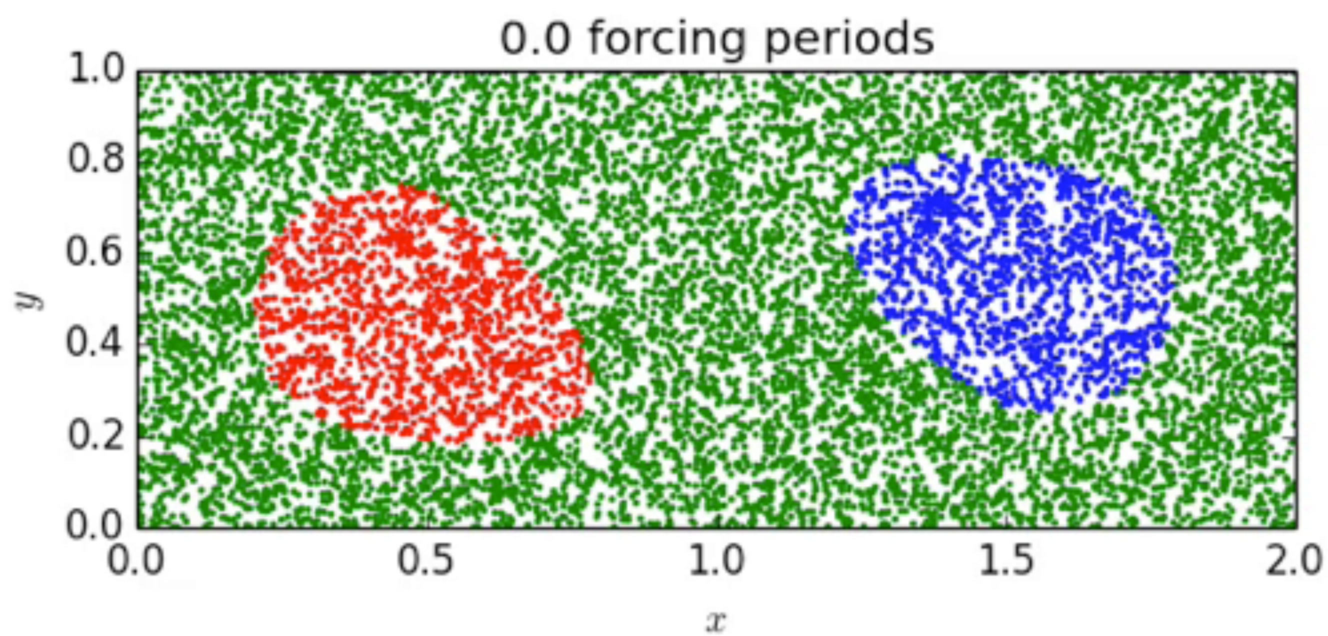
Second eigenfunction



Third eigenfunction

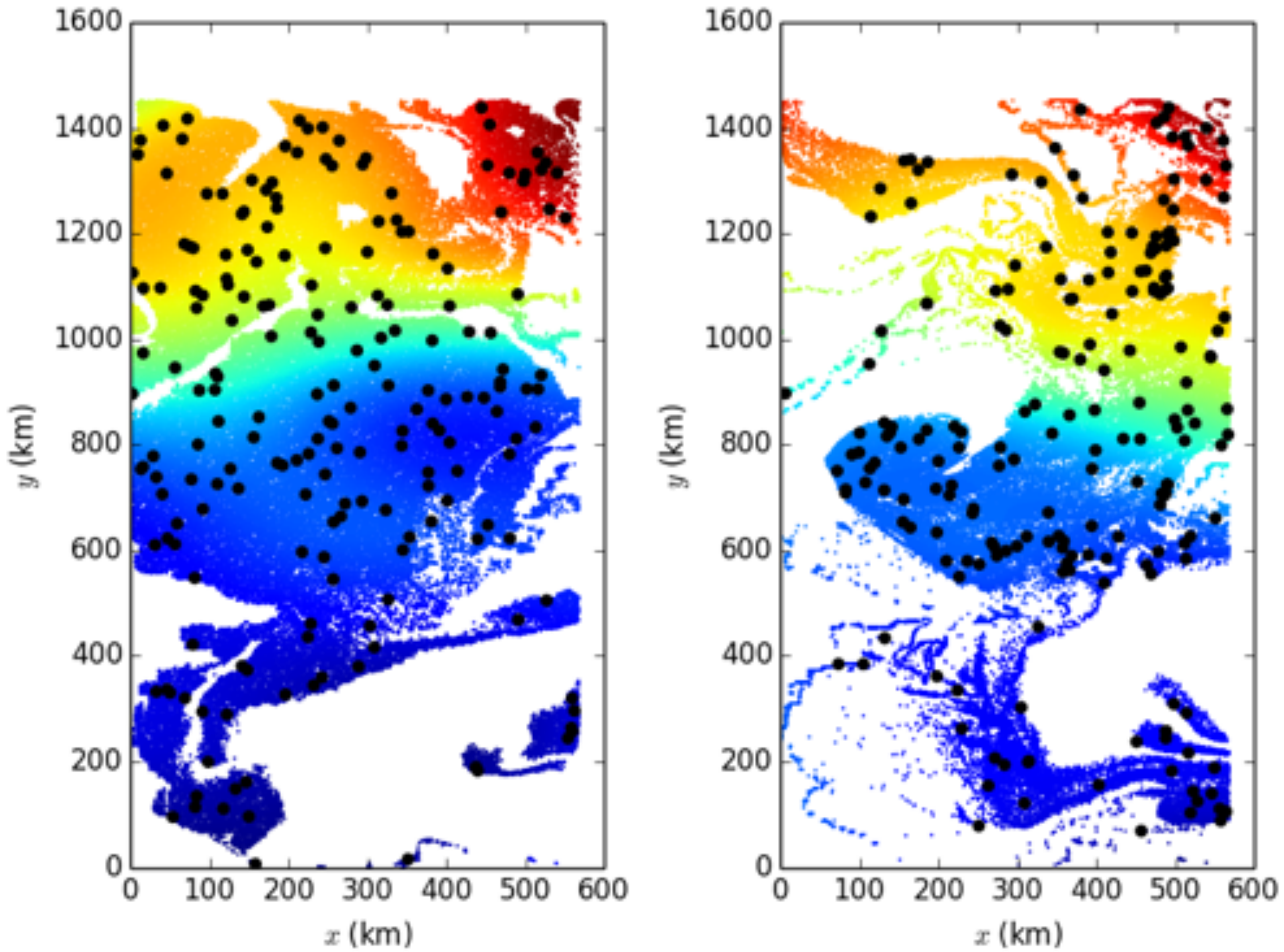


Almost invariant sets



Coherent sets in the Philippine Sea

- Compute coherent sets from ocean drifter data
  - Simulated drifter trajectories generated from a numerical model for the velocity field near the Philippines (with I. Rypina, WHOI)

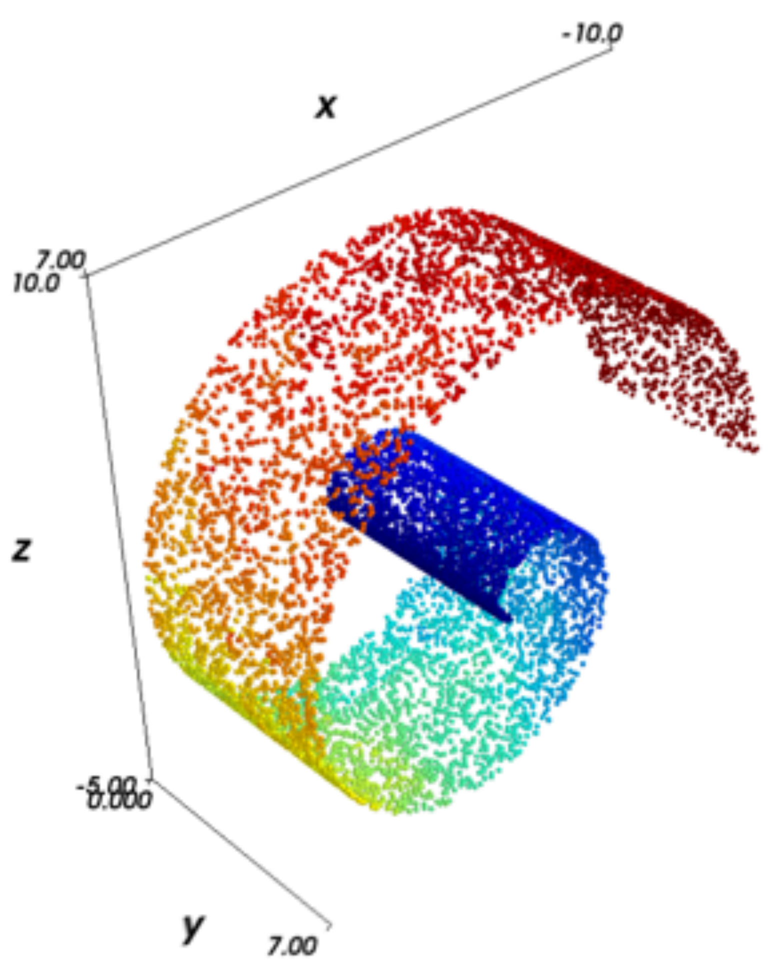


  - 20 radial basis functions, and 200 data points (black dots)
  - *k*-means clustering used for RBF centers
  - 10^5 points added for visualization

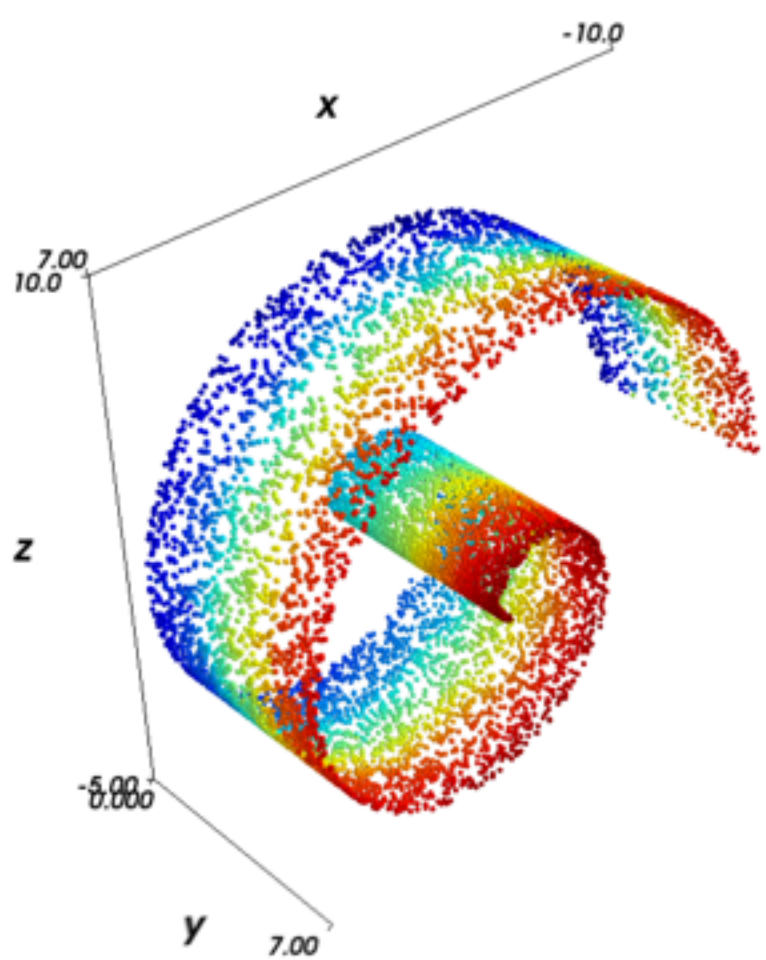
Determining coordinates for nonlinear systems

- Koopman eigenfunctions can be used to provide useful coordinates for nonlinear systems (similar to diffusion maps)
- Example: diffusion process on a "swiss roll"
- First two Koopman eigenfunctions extract "length" and "width"
- Diffusion maps extract only geometry
  Koopman extracts geometry and dynamics

First eigenfunction



Second eigenfunction



Summary

- The CDS community has had a major impact on fluid mechanics
  - Flow control
  - Coherent structures
- Recent themes:
  - Analysis based on data
  - Finite-time

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