Lower bounds on semidefinite representations

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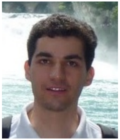

Based on joint work
with **Hamza Fawzi** and **James Saunderson**

CDS and Optimization

Why optimization?

The underlying machinery for much of CDS:

- Robustness analysis (small gain, μ, IQCs, SOS, etc.) – Andy's talk
- MPC, on-line optimization – Manfred's talk
- Variational principles

Control = optimization + (temporal) structure

Goal: decouple optimization difficulties (nonlinearity, nonconvexity, etc.) from dynamical aspects.

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Question: representability of convex sets

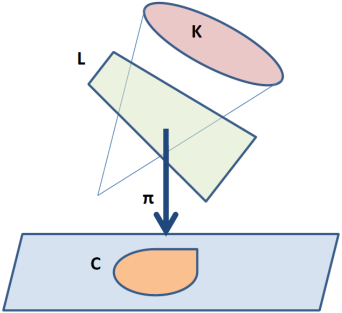
Existence and efficiency:

- When is a convex set representable by conic optimization?
- How to quantify the number of additional variables that are needed?

Given a convex set C, is it possible to represent it as

$$C = \pi(K \cap L)$$

where K is a (fixed) cone, L is an affine subspace, and π is a linear map?



"Extended formulations" and polytopes

What happens in the case of polytopes?

$$P = \{x \in \mathbb{R}^n \, : \, {f_i}^T x \leq 1\}$$

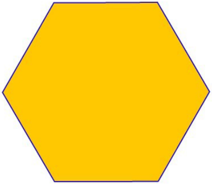
(WLOG, compact with $0 \in \operatorname{int} P$).

Polytopes have a finite number of facets f_i and vertices v_j.
Define a nonnegative matrix, called the *slack matrix* of the polytope:

$$[S_P]_{ij} = 1 - {f_i}^T v_j, \qquad i = 1, \dots, |F| \quad j = 1, \dots, |V|$$

Example: hexagon (I)

Consider a regular hexagon in the plane.



It has 6 vertices, and 6 facets. Its slack matrix has rank 3, and is

$$S_H = \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

"Trivial" representation requires 6 facets. Can we do better?

Cone factorizations and representability

"Geometric" LP formulations exactly correspond to "algebraic" factorizations of the slack matrix.

For polytopes, this amounts to a *nonnegative factorization* of the slack matrix:

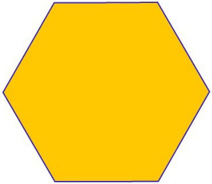
$$S_{ij} = \langle a_i, b_j \rangle, \qquad i = 1, \ldots, v, \qquad j = 1, \ldots, f$$

where a_i, b_i are nonnegative vectors.

Theorem (Yannakakis 1991): The minimal lifting dimension of a polytope is equal to the *nonnegative rank* of its slack matrix.

Example: hexagon (II)

Regular hexagon in the plane.



Its slack matrix is

$$S_H = \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

Nonnegative rank is 5.

Beyond LPs and nonnegative factorizations

LPs are nice, but what about broader representability questions?

In [GPT11], a generalization of Yannakakis' theorem to the general convex case. General theme:

"Geometric" extended formulations exactly correspond to "algebraic" factorizations of a slack operator.

polytopes/LP	convex sets/convex cones
slack matrix	slack operators
vertices	extreme points of C
facets	extreme points of polar C°
nonnegative factorizations	conic factorizations
$S_{ij} = \langle a_i, b_j \rangle, \quad a_i \geq 0, b_j \geq 0$	$S_{ij} = \langle a_i, b_j \rangle, \quad a_i \in K, b_j \in K^*$

Polytopes, semidefinite programming, and factorizations

Even for polytopes, PSD factorizations can be interesting.

Well-known example: the *stable set* or *independent set* polytope.

For perfect graphs, we have efficient SDP representations, but no known subexponential LP.

Natural notion: *positive semidefinite rank* ([GPT 11]).
Exactly captures the complexity of SDP-representability.

PSD rank of a nonnegative matrix

Let $M \in \mathbb{R}^{m \times n}$ be a nonnegative matrix.

Definition [GPT11]: The *PSD rank* of M, denoted rank_{psd}, is the smallest r for which there exists $r \times r$ PSD matrices $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_n\}$ such that

$$M_{ij} = \text{trace}\, A_i B_j, \qquad i = 1, \ldots, m \quad j = 1, \ldots, n.$$

Natural generalization of nonnegative rank. "Quantum" analogue of nonnegative factorizations.

PSD rank of slack matrix determines the "best" semidefinite lifting.

E.g., for the regular hexagon, PSD rank is 4.

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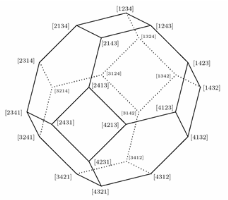
Orbitopes and equivariant lifts

Special class of convex bodies: regular orbitopes

$$C = \{\mathrm{conv}(g \cdot x_0) \,:\, g \in G\},$$

where G is a compact group.



Many important examples: hypercubes, hyperspheres, nuclear norm, Grassmannians, Birkhoff polytope, parity polytope, cut polytope, ...

For many reasons, important to look at "symmetric" (or equivariant) lifts. Informally, the lift "respects" the symmetries of the convex body C. Formally, there is a group homomorphism $\rho : G \to GL(\mathbb{R}^d)$ such that

- L is invariant under conjugation by ρ,
- ρ "intertwines" the lift map: $\pi(\rho(T) Y \rho(T)^T) = T\pi(Y), \qquad \forall T \in G, \forall Y \in \mathcal{S}_+^d \cap L.$

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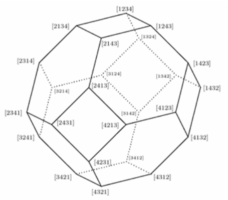

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A structure theorem for equivariant lifts

Equivariant lifts of orbitopes are particularly nice.

Why?: Every *equivariant* SDP lift is of *sum of squares* type.

More formally:

Theorem [FSP 13]: Let P be a G-regular orbitope, with a G-equivariant lift of size d. Then for any linear form ℓ nonnegative on P, there exist functions $f_j \in V$ such that

$$\ell_{\max} - \ell(x) = \sum_j f_j(x)^2 \qquad \forall x \in X$$

where $X = \text{ext}(P)$, and V is a G-invariant subspace of $\mathcal{F}(X)$.

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Lower bounding size of representations

Why is this useful?

Can use *representation theory* to understand invariant subspaces of $\mathcal{F}(X)$.

For polytopes, these are finite-dimensional subspaces of polynomials. Computing their dimensions, we obtain lower bounds on symmetric representations.

Parity and cut polytopes

The *parity polytope* PAR_n is the convex hull of all points $x \in \{-1, 1\}^n$ that have an even number of -1.

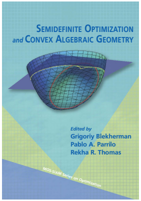
Theorem [FSP13]: Let PAR_n be the parity polytope. Then, any Γ_{parity}-equivariant psd lift of PAR_n must have size $\geq \binom{n}{\lceil n/4 \rceil}$.

The *cut polytope* is defined as

$$CUT_n = \text{conv}(xx^T \, : \, x \in \{-1, 1\}^n).$$

Theorem [FSP13]: Any psd lift of CUT_n that is equivariant with respect to the cube (hyperoctahedral) group must have size $\geq \binom{n}{\lceil n/4 \rceil}$.

The End

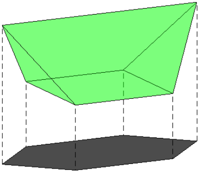


Thank You!

Want to know more?

- G. Blekherman, P.A. Parrilo, R. Thomas (eds.), *Semidefinite Optimization and Convex Algebraic Geometry*, SIAM, 2013.
- J. Gouveia, P.A. Parrilo, R. Thomas, Lifts of convex sets and cone factorizations, *Mathematics of Operations Research*, 38:2, 2013.
- J. Gouveia, P.A. Parrilo, R. Thomas, Approximate cone factorizations and lifts of polytopes, `arXiv:1308.2162`.
- H. Fawzi, P.A. Parrilo, Exponential lower bounds on fixed-size psd rank and semidefinite extension complexity, `arXiv:1311.2571`.
- H. Fawzi, J. Saunderson, P.A. Parrilo, Equivariant semidefinite lifts and sum-of-squares hierarchies, `arXiv:1312.6662`.
- H. Fawzi, J. Gouveia, P.A. Parrilo, R. Robinson, R. Thomas, Positive semidefinite rank, `arXiv:1407.4095`.

Example: hexagon (III)



A nonnegative factorization:

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