Fundamentals of Control

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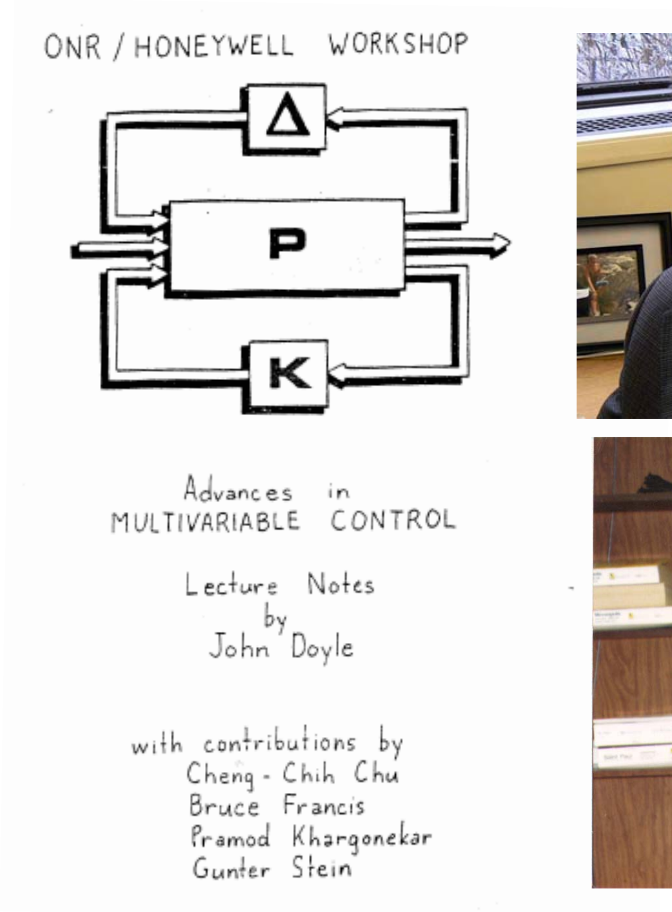
Past



Gary Balas

Roy Smith

Caltech connection:

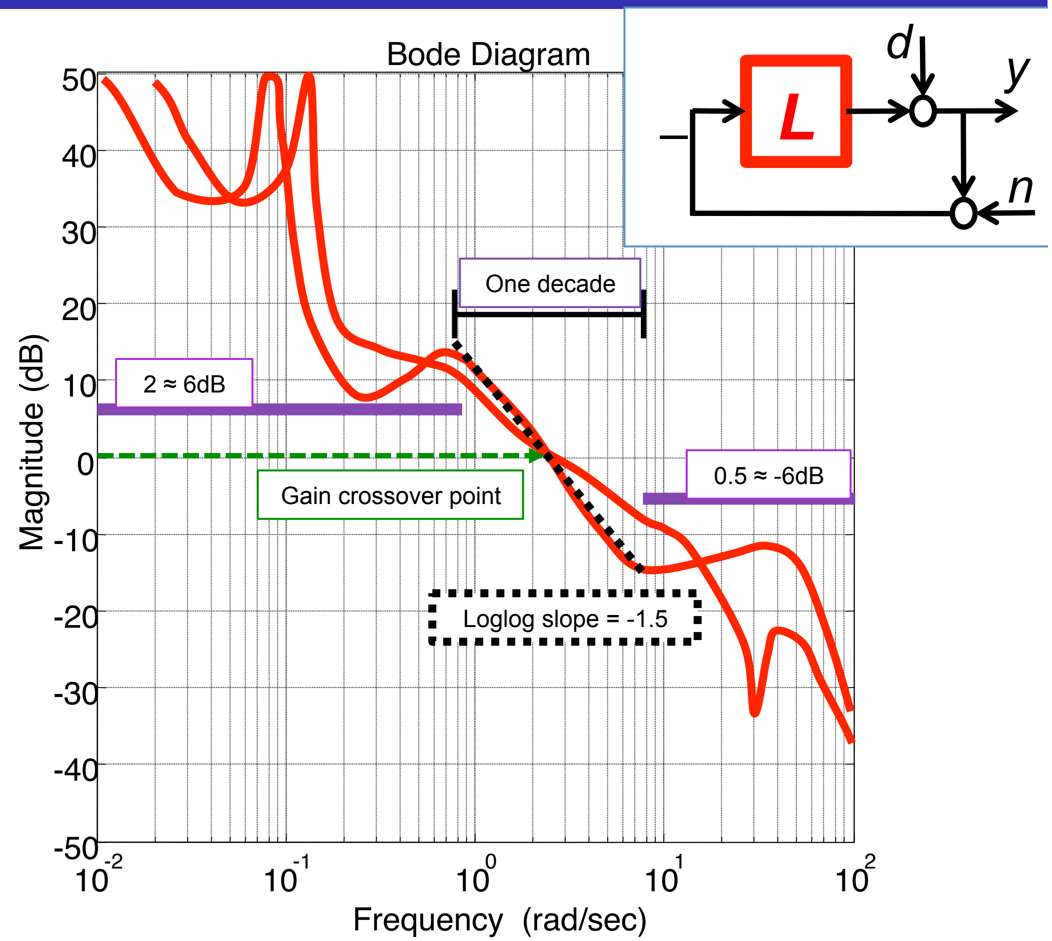
- Visiting grad student in EE at various times in late 1986, 1987
- Post-doc in EE, 1988
- Classmate of Richard (1985-87)

SISO design: generalize neg feedback around integrator is good

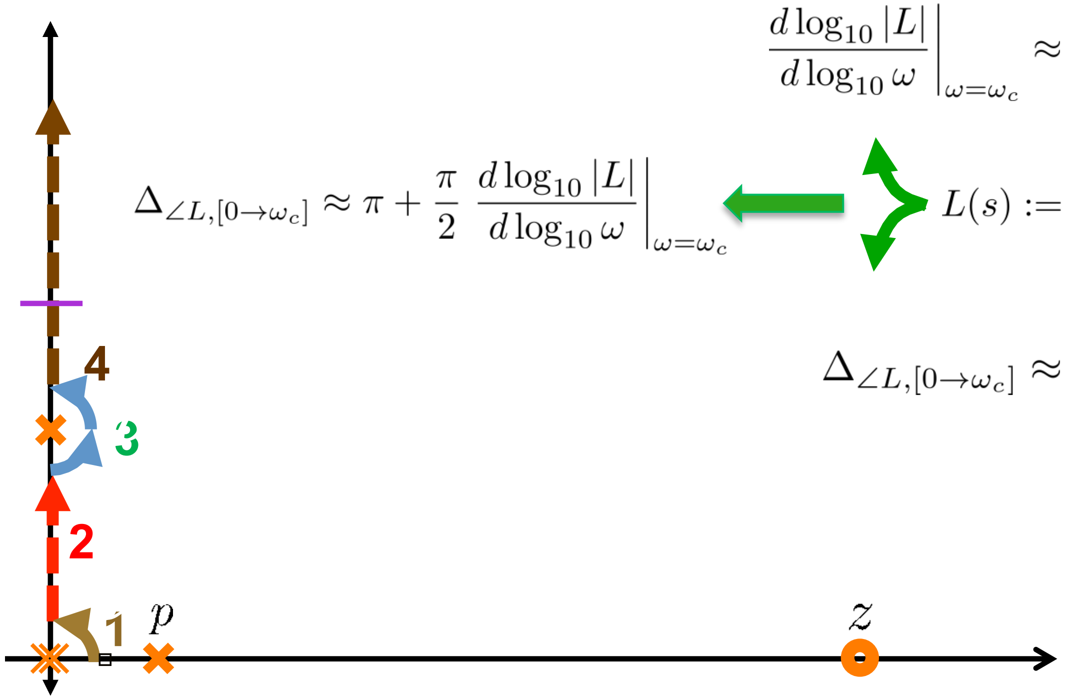
Suppose $H(s)$ has no poles or zeros in CRHP. Assume

- $p>0$, $z>0$, $H(0)>0$; and
$$L(s) = \frac{1}{s^2+\omega_R^2}\frac{-s+z}{s^m(s-p)}H(s)$$
- $L(s)$ has one gain-crossover point, ω_c, and $3p < \omega_c < 1/3\, z$, and $\omega_c > \omega_R$
- The loglog slope of $|L|$ satisfies
$$0 > \left.\frac{d\log_{10}|L|}{d\log_{10}\omega}\right|_{\omega\in\left[\frac{\omega_c}{\sqrt{10}},\sqrt{10}\omega_c\right]} \geq -1.5$$
- Outside this interval, the magnitude is
  - greater than 2, or less than 0.5

Then, the closed-loop system is stable and has modest phase (and gain) margins, and the peak of $|S|$ is less than 2.5



Proof: Nyquist and Bode-Phase theorem



$$\left.\frac{d\log_{10}|L|}{d\log_{10}\omega}\right|_{\omega=\omega_c} \approx \quad -2 \quad -m \quad 0 \quad -1 \quad \left.\frac{d\log_{10}|H|}{d\log_{10}\omega}\right|_{\omega=\omega_c}$$

$$\Delta_{\angle L,[0\to\omega_c]} \approx \pi + \frac{\pi}{2}\left.\frac{d\log_{10}|L|}{d\log_{10}\omega}\right|_{\omega=\omega_c}$$

$$L(s) := \frac{1}{s^2+\omega_R^2} \quad \frac{1}{s^m} \quad (-s+z) \quad \frac{1}{s-p} \quad H(s)$$

$$\Delta_{\angle L,[0\to\omega_c]} \approx \quad -\pi \quad -m\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \Delta_{\angle H,[0\to\omega_c]}$$

$$\approx \frac{\pi}{2}\left.\frac{d\log_{10}|H|}{d\log_{10}\omega}\right|_{\omega=\omega_c}$$

Now use assumption:

- The loglog slope of $|L|$ satisfies

$$0 > \left.\frac{d\log_{10}|L|}{d\log_{10}\omega}\right|_{\omega\in\left[\frac{\omega_c}{\sqrt{10}},\sqrt{10}\omega_c\right]} \geq -1.5$$

$$\frac{\pi}{4} \leq \Delta_{\angle L,[0\to\omega_c]} \leq \pi$$

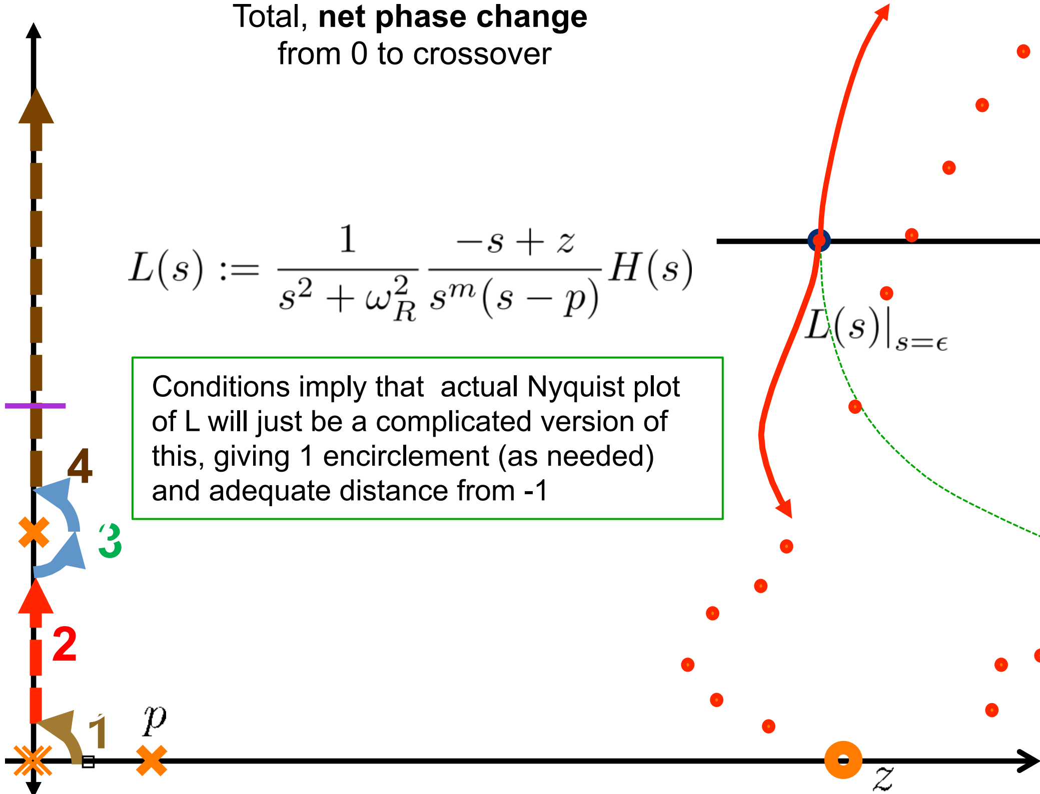
Graphical Interpretation

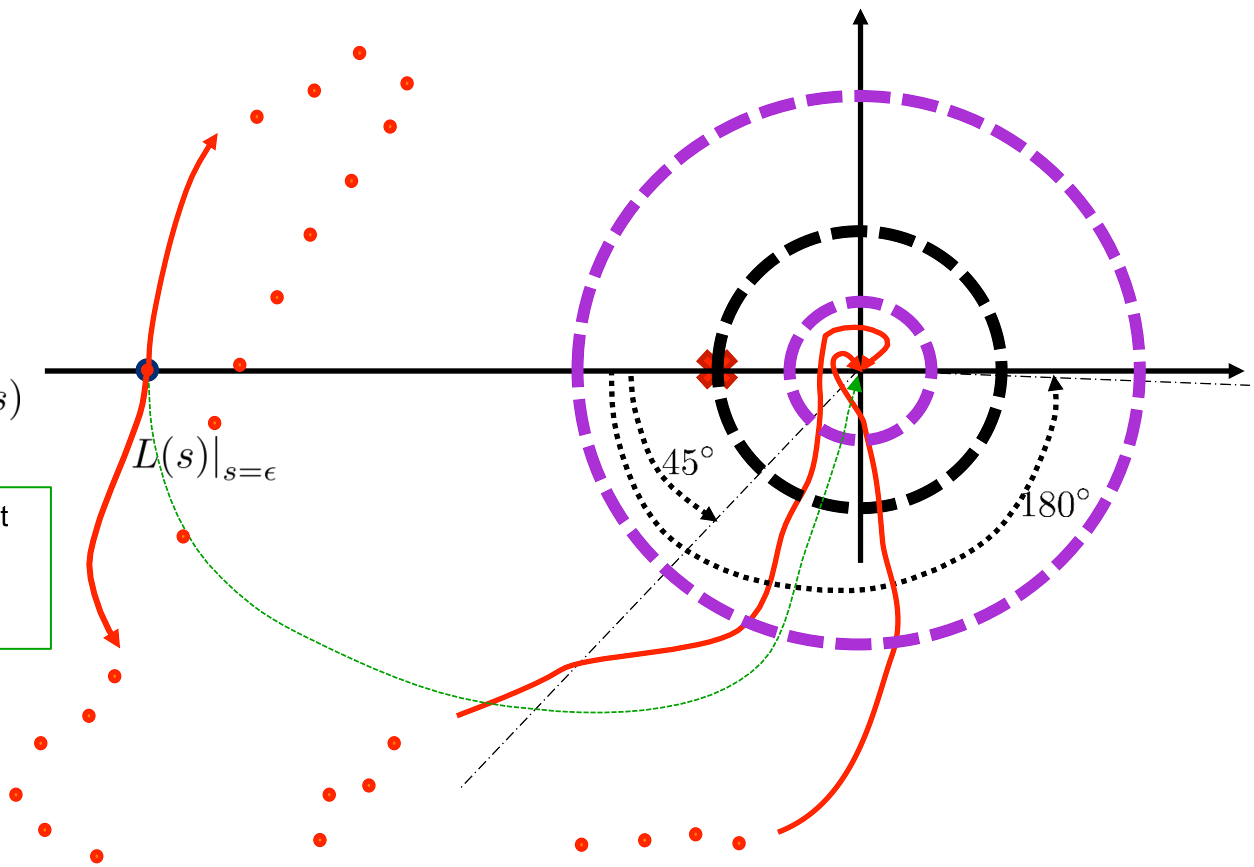
$$\frac{\pi}{4} \leq \Delta_{\angle L,[0\to\omega_c]} \leq \pi$$

Total, **net phase change** from 0 to crossover

$$L(s) := \frac{1}{s^2+\omega_R^2}\frac{-s+z}{s^m(s-p)}H(s)$$

Conditions imply that actual Nyquist plot of L will just be a complicated version of this, giving 1 encirclement (as needed) and adequate distance from -1





Prototypical robustness analysis by control community

Components

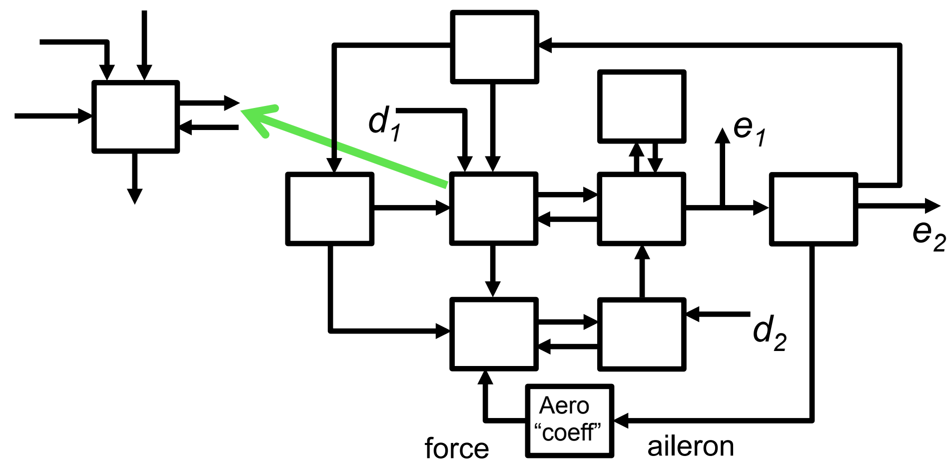
- Relations among variables

External variables (*d*)

Selected internal variables (*e*)

Interconnection

- Equates variables of "communicating" components
- Implicitly gives (*d*/*e*) relation



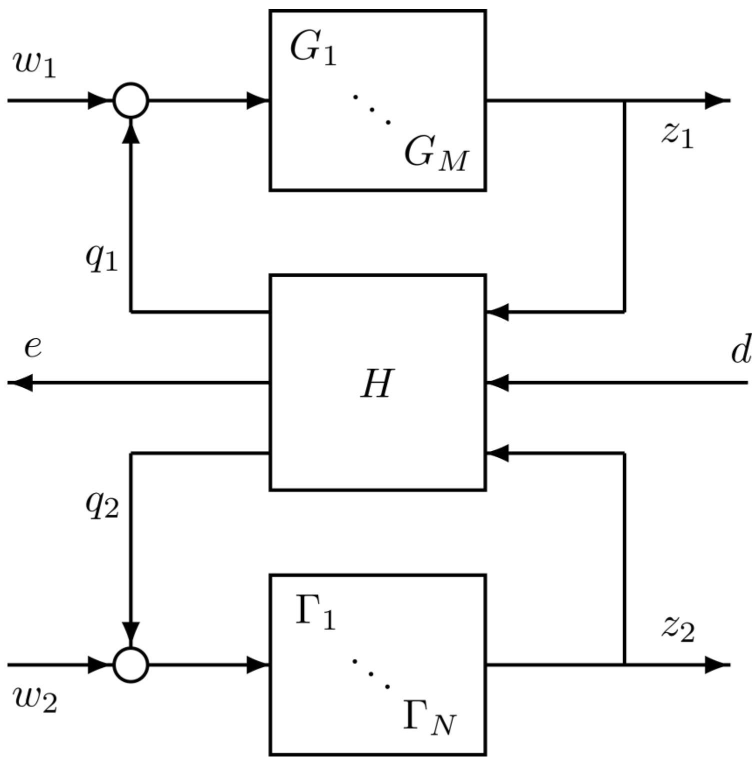
How is component uncertainty quantified?

- List of quadratic (in)equalities that variables it relates are **guaranteed** to satisfy
- "Certain": just a special case of uncertain
- Uncertainty in (*d/e*) is quantified in same manner – certify that (*d/e*) relation always satisfies specific quadratic inequalities

Robustness question

- Uncertain components
  - Uncertainty is quantified at component level
- Quantify uncertainty in (*d*/*e*) relation

LTI version, with certain G and uncertain Γ



– each FDLTI, with proper transfer function, and stabilizable/detectable state-space description

– constant interconnection matrix

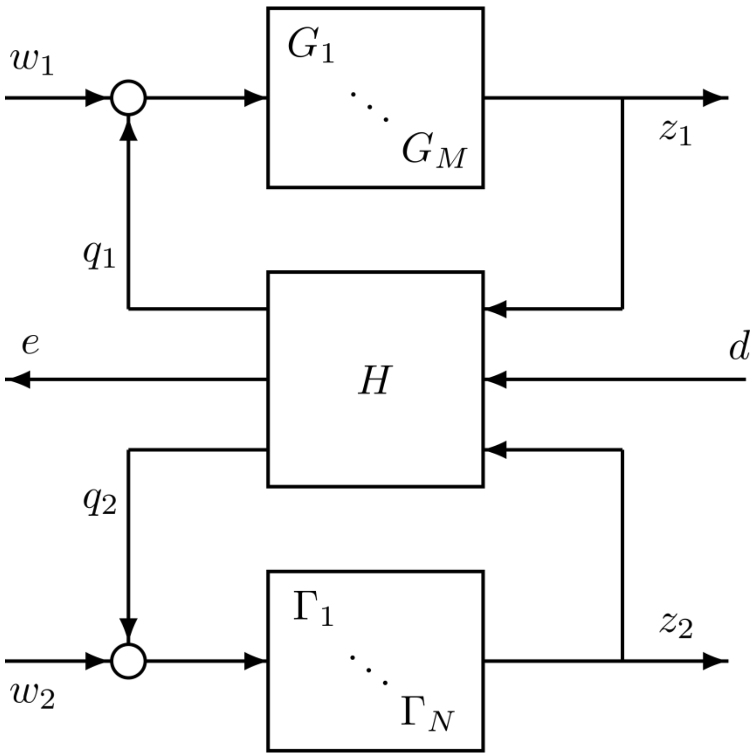
– *well-posed*: for any initial conditions and any piecewise-continuous inputs w_1, w_2, d, there exist unique solutions to the interconnection

- For a well-posed interconnection, a state-space model or proper transfer function description for the map from (d,w) to (e,z) can be derived.

– *stable* if the resultant state-space model is internally stable – eigenvalues of it's "A" matrix are in the open, left-half plane

– If stable, what is gain from $d \rightarrow e$

Answers, separate from uncertain analysis



Well-posed if and only if

$$\det\left(I - \begin{bmatrix} H_{11} & H_{13} \\ H_{31} & H_{33} \end{bmatrix}\begin{bmatrix} G(\infty) & 0 \\ 0 & \Gamma(\infty) \end{bmatrix}\right) \neq 0$$

$$\Leftrightarrow T_{wz} \in \mathcal{R}^{\bullet\times\bullet}$$

Stable if and only if

$$T_{wz} \in \mathbb{R}H_\infty^{\bullet\times\bullet}$$

Quantify gain

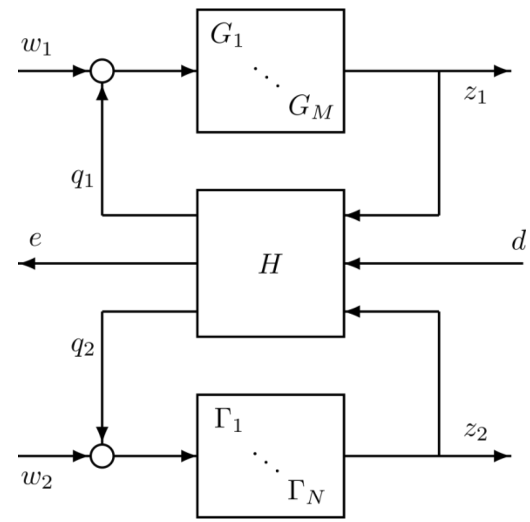
$$\|T_{de}\|_\infty$$

Simplest assumptions on unknown elements

1. Γ_k is a stable linear system, known only to satisfy $\|\Gamma_k\|_\infty < 1$;
2. Γ_k is a stable linear system of the form $\gamma_k I$, where the scalar linear system γ_k is known to satisfy $\|\gamma_k\|_\infty < 1$;
3. Γ_k is a constant gain, of the form $\gamma_k I$, where the scalar $\gamma_k \in \mathbf{R}$ is known to satisfy $-1 < \gamma_k < 1$.

Is the interconnection well-posed and stable for all possible values of Γ?

If so, is the $\|\cdot\|_\infty$ gain from $d \to e$ ≤ 1 for all possible values of Γ?



Interconnection: robust well-posedness and stability

Interconnection is well-posed at

$$\det\,(I - G(\infty)H_{11}) \neq 0$$

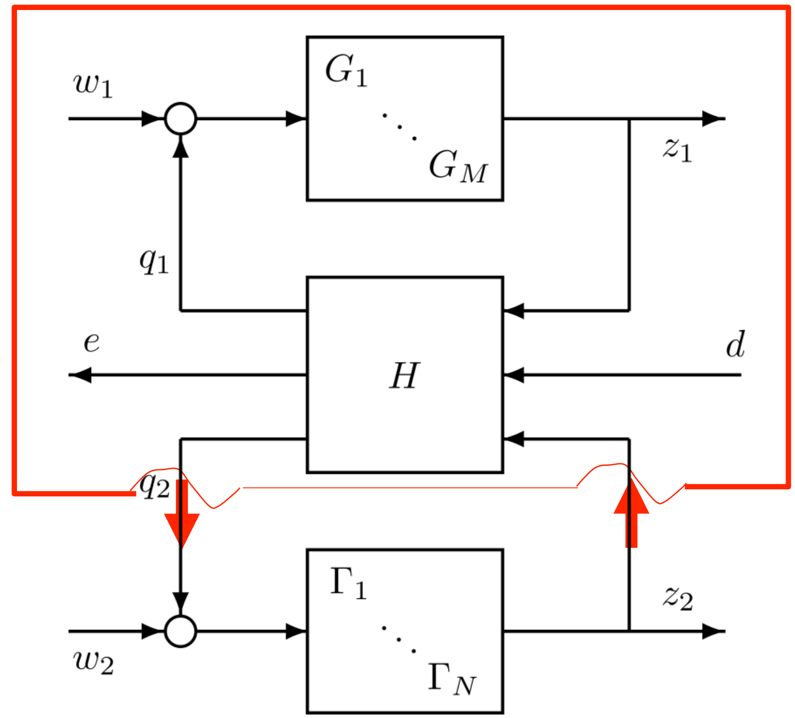
$$V := G(s)(I - H_{11}G(s))^{-1} \in \mathcal{R}^{\bullet\times\bullet}$$

Interconnection is stable at

$$V := G(s)(I - H_{11}G(s))^{-1} \in \mathbb{R}H_\infty^{\bullet\times\bullet}$$

$$M := H_{33} + H_{31}VH_{13} \in \mathbb{R}H_\infty^{\bullet\times\bullet}$$

$$X := I - \Gamma M$$



Interconnection is well-posed at

$$\det\,(I - \Gamma(\infty)M(\infty)) = \det\,(X(\infty)) \neq 0.$$

$$X^{-1} \text{ is proper}$$

Interconnection is stable at

$$\det\,(I - \Gamma(s_0)M(s_0)) = \det\,(X(s_0)) \neq 0 \quad \forall s_0 \in \mathbf{C}_+$$

$$X^{-1} \in \mathbb{R}H_\infty^{\bullet\times\bullet}$$

Non-vanishing determinant conditions

Complex numbers mimicking dynamic systems

Theorem: Given a positive $\bar{\omega} > 0$, and a complex number δ, with $\text{Imag}(\delta) \neq 0$, there is a $\beta > 0$ such that by proper choice of sign

$$\pm |\delta| \left. \frac{s-\beta}{s+\beta} \right|_{s=j\bar{\omega}} = \delta$$

Given $\bar{\omega} > 0$ and $G \in \mathbb{R}H_\infty^{n\times n}$:

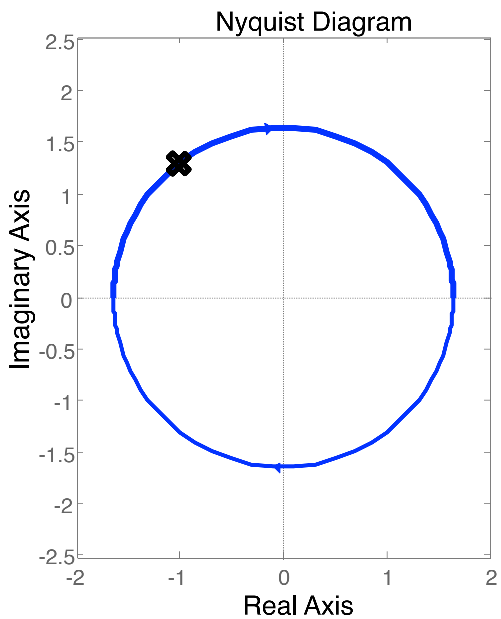
$\exists \Delta \in \mathbb{C}^{n\times n}$, $\bar{\sigma}(\Delta) \leq \alpha$ with

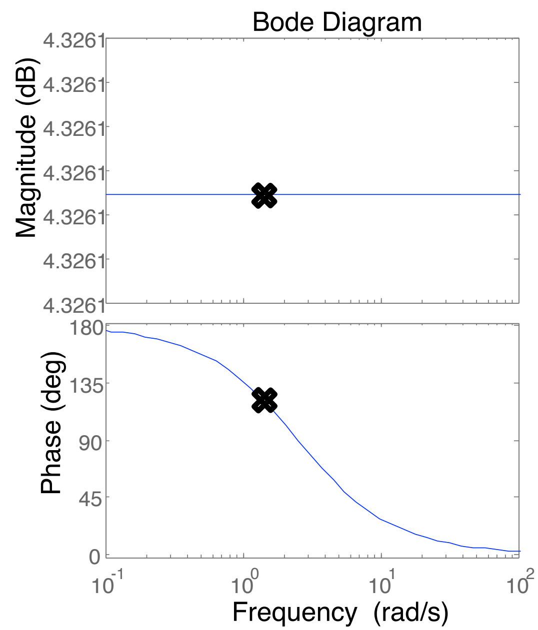
$$\det(I - G(j\bar{\omega})\Delta) = 0$$

$\Updownarrow$

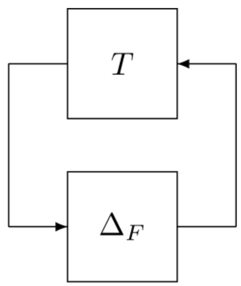
$\exists\, \Gamma \in \mathbb{R}H_\infty^{n\times n}$, $\|\Gamma\|_\infty \leq \alpha$ with

$$\det(I - G(j\bar{\omega})\Gamma(j\bar{\omega})) = 0$$

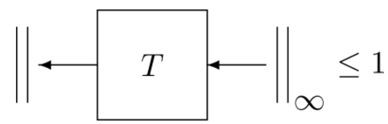


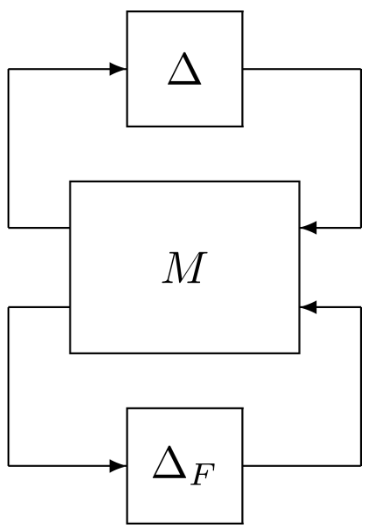


Gain as robustness

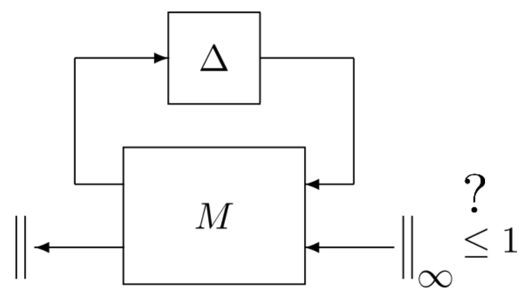


Stable, well-posed for all $\|\Delta_F\|_\infty < 1$





formulate as



Definition

Importance of the nonvanishing determinant condition

- new definition to formalize,
- separate arithmetic from system theory.

J. Doyle, "Analysis of feedback systems with structured uncertainties," *IEE Proceedings*, part D, vol. 129, no. 6, pp. 242-250, 1982.

Example: a problem-specific set of block diagonal matrices, say,

$$\mathbf{\Delta} := \left\{ \operatorname{diag}\left[\delta^r I, \delta^c I, \Delta_F\right] : \delta^r \in \mathbf{R}, \delta^c \in \mathbf{C}, \Delta_F \in \mathbf{C}^{f \times f} \right\} \subseteq \mathbf{C}^{n \times n}$$

For $M \in \mathbf{C}^{n \times n}$ define

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\right\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $(I - M\Delta)$ singular, then $\mu_{\mathbf{\Delta}}(M) := 0$.

Definition

A general $\mathbf{\Delta} \subseteq \mathbf{C}^{n \times n}$ is of the same form

$$\mathbf{\Delta} := \left\{ \text{diag}\left[\delta^r I_t, \delta^c I_v, \Delta_F\right] : \delta^r \in \mathbf{R}, \delta^c \in \mathbf{C}, \Delta_F \in \mathbf{C}^{f \times f} \right\} \subseteq \mathbf{C}^{n \times n}$$

but likely include many instances of the 3 elements considered.

$$\boxed{\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\right\}}}$$

- $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n \times n} \to \mathbf{R}$
- Smallest (measured in $\bar{\sigma}(\cdot)$) root, drawn from $\mathbf{\Delta}$, of the polynomial equation $\det(I - M\Delta) = 0$
- For any $\alpha \in \mathbf{R}$, $\mu(\alpha M) = |\alpha| \mu(M)$
- $\mu_{\mathbf{\Delta}}(M) < 1$ iff $\det(I - M\Delta) \neq 0 \;\; \forall \Delta \in \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\} =: \mathbf{B}_{\mathbf{\Delta}}$

Maximum-modulus

For $M \in \mathbb{R}H_\infty^{n\times n}$, and any block-structure $\mathbf{\Delta}$,

$$\max\left\{\sup_{\mathrm{Re}(s)\geq 0} \mu_{\mathbf{\Delta}}(M(s)) \ , \ \mu_{\mathbf{\Delta}}(M_\infty)\right\} = \max\left\{\sup_{\omega\in\mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \ , \ \mu_{\mathbf{\Delta}}(M_\infty)\right\}$$

Continuity

$\mu_{\mathbf{\Delta}} : \mathbf{C}^{n\times n} \to \mathbf{R}$ is upper-semicontinuous, but (in general) not continuous

- If $\mathbf{\Delta}$ only consists of complex blocks, then $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n\times n} \to \mathbf{R}$ is continuous
- If $\mathbf{\Delta}$ has no repeated-reals, then $\mu_{\mathbf{\Delta}} : \mathbf{R}^{n\times n} \to \mathbf{R}$ is continuous
- Suppose $\mathbf{\Delta}$ is a diagonal concatenation of uncertainty sets, one with only real blocks, and one with only complex blocks. Denote these as $\mathbf{\Delta_R}$ and $\mathbf{\Delta_C}$. So

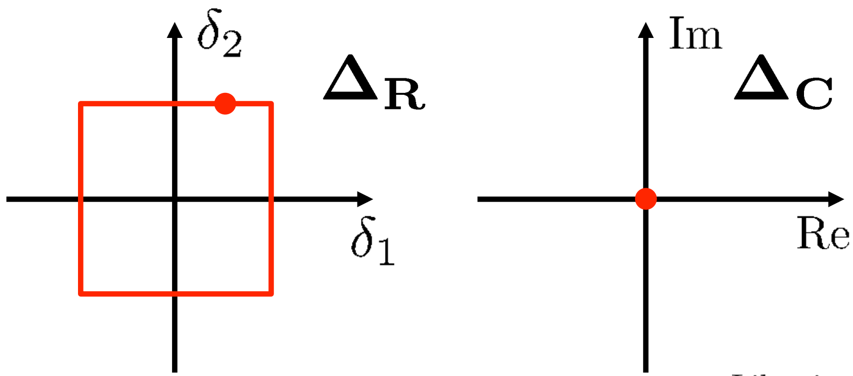
$$\mathbf{\Delta} = \{\mathrm{diag}\,[\Delta_R, \Delta_C] : \Delta_R \in \mathbf{\Delta_R}, \Delta_C \in \mathbf{\Delta_C}\} \subseteq \mathbf{C}^{n\times n}$$

$$\det\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \Delta_R & 0 \\ 0 & 0 \end{bmatrix}\right) = \det\left(I - M_{11}\Delta_R\right)$$

$$\det\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \Delta_R & 0 \\ 0 & \Delta_C \end{bmatrix}\right) = \det\left(I - M\Delta\right)$$

If $\mu_{\mathbf{\Delta_R}}(M_{11}) < \mu_{\mathbf{\Delta}}(M)$, then $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n\times n} \to \mathbf{R}$ is continuous at M.

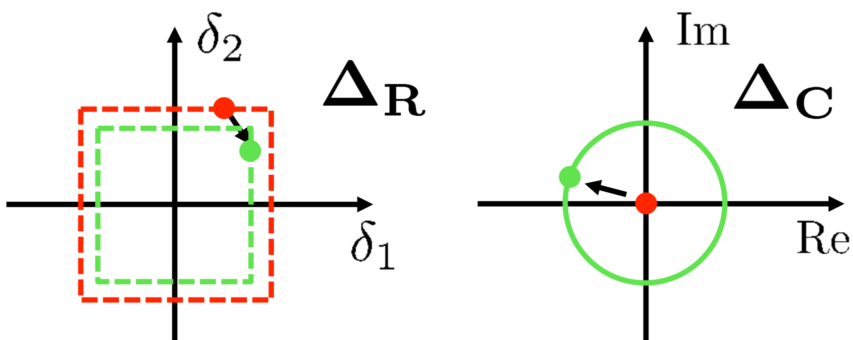
Guaranteeing continuity



Create singularity using
<u>only</u> Δ_R (with $\Delta_C := 0$)

$$\det\left(\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} - \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\boxed{\begin{bmatrix}\Delta_R & 0\\ 0 & 0\end{bmatrix}}\right) = \det\left(I - M_{11}\Delta_R\right)$$

Likewise, if $\mu_{\boldsymbol{\Delta}}$ is not continuous at M, then $\mu_{\boldsymbol{\Delta}_{\mathbf{R}}}(M_{11}) = \mu_{\boldsymbol{\Delta}}(M)$. This means "the complex blocks do not matter" and hence can be set to 0.



There are complex blocks, and they "matter"

Create singularity using
<u>both</u> Δ_R and Δ_C

$$\det\left(\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} - \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\boxed{\begin{bmatrix}\Delta_R & 0\\ 0 & \Delta_C\end{bmatrix}}\right)$$

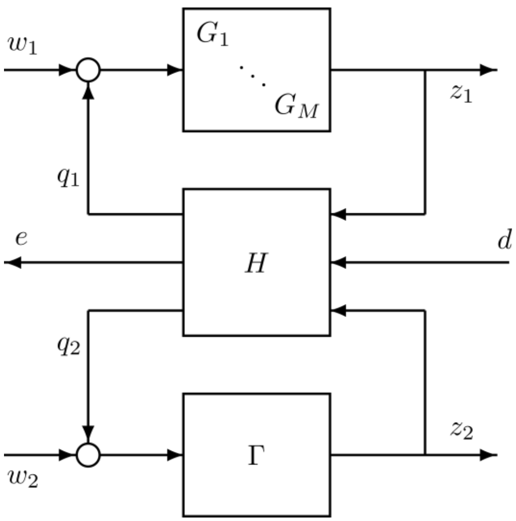
Robustness "test"

- $\boldsymbol{\Delta} \subseteq \mathbf{C}^{m \times n}$, and associated $\boldsymbol{\Gamma}$

$$\boldsymbol{\Delta} = \{\mathrm{diag}\left[\delta_1^r I_{t_1}, \ldots, \delta_V^r I_{t_V}, \delta_1^c I_{r_1}, \ldots, \delta_S^c I_{r_S}, \Delta_1, \ldots, \Delta_F\right] : \\ \delta_k^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \Delta_j \in \mathbf{C}^{m_j \times n_j}\}$$

$$\boldsymbol{\Gamma} := \{\mathrm{diag}\left[\gamma_1^r I_{t_1}, \ldots, \gamma_V^r I_{t_V}, \gamma_1(s) I_{r_1}, \ldots, \gamma_S(s) I_{r_S}, \Gamma_1(s), \ldots, \Gamma_F(s)\right] : \\ \gamma_k^r \in \mathbf{R}, \gamma_i \in \mathcal{S}, \Gamma_j \in \mathcal{S}^{m_j \times n_j}\}$$

- Partial knowledge is $\Gamma \in \boldsymbol{\Gamma}$ and $\|\Gamma\|_\infty < 1$

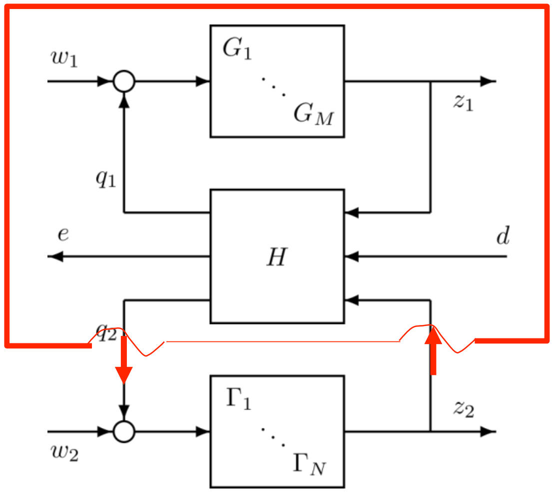


Interconnection is stable at Γ

$$\det\left(I - \Gamma(s_0) M(s_0)\right) = \det\left(X(s_0)\right) \neq 0 \quad \forall s_0 \in \mathbf{C}_+$$

Complex mimic dynamics
Maximum-modulus thm

Robustness "test" and stability margin



Theorem: (G, H, Γ) interconnection is well-posed and stable for all $\Gamma \in \mathbf{\Gamma}$ with $\|\Gamma\|_\infty < \frac{1}{\beta}$ if and only if

$$M \in \mathbb{R}H_\infty^{n\times n} \text{ and}$$

$$\max_{\omega \in \mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega)) := \max\left\{\sup_{\omega\in\mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \;,\; \mu_{\mathbf{\Delta}}(M_\infty)\right\} \leq \beta$$

$$\text{Stability Radius} = \frac{1}{\max_{\omega\in\mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega))}$$

Characterizing as constraint implication

$\det(I - M\Delta) \neq 0$ if and only if

$$\begin{aligned} z &= Mw \\ w &= \Delta z \end{aligned} \quad \Rightarrow \quad w = 0$$

$\mu_{\boldsymbol{\Delta}}(M) < 1$ if and only if

$$\boxed{\begin{aligned} z &= Mw \\ w &= \Delta z \\ \Delta &\in \mathbf{B}_{\boldsymbol{\Delta}} \end{aligned} \quad \Rightarrow \quad w = 0}$$

Computable conditions which certify this implication are "upper bound" methods

$w=\Delta z,\ \Delta\in\mathbf{B}{\downarrow}\Delta$ as quadratic constraints on (z, w)

Theorem: Given $z \in \mathbf{C}^n$ and $w \in \mathbf{C}^n$. There exists $\delta \in \mathbf{C}$, with $|\delta| \leq 1$ and $w = \delta z$ if and only if $zz^* - ww^* \succeq 0$

Theorem: Given $z \in \mathbf{C}^n$ and $w \in \mathbf{C}^n$. There exists $\delta \in \mathbf{R}$, with $|\delta| \leq 1$ and $w = \delta z$ if and only if

$$zz^* - ww^* \succeq 0, \qquad zw^* - wz^* = 0$$

Theorem: Given $z \in \mathbf{C}^n$ and $w \in \mathbf{C}^m$. There exists $\Delta \in \mathbf{C}^{m\times n}$ with $\bar{\sigma}\left(\Delta\right) \leq 1$ and $w = \Delta z$ if and only if

$$z^*z - w^*w \geq 0$$

Bounding μ: Inequalties implying another inequality

Canonical $\mathbf{\Delta}$ (general $\mathbf{\Delta}$ just has more of each type)

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

Theorem: Given $z, w \in \mathbf{C}^n$. There exists $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ such that $w = \Delta z$ and $z = Mw$ if and only if (M_i and E_i are appropriate rows of M and Identity)

1. $M_1 ww^* M_1^* - E_1 ww^* E_1^* \succeq 0, \quad M_1 ww^* E_1^* - E_1 ww^* M_1^* = 0,$
2. $M_2 ww^* M_2^* - E_2 ww^* E_2^* \succeq 0$
3. $w^* \left(M_3^* M_3 - E_3^* E_3 \right) w \geq 0.$

When do these imply $w = 0$?
...proving $\mu_{\mathbf{\Delta}}(M) < 1$.

S-procedure: implications, containments, empty intersections

$$\begin{array}{l} h_1(\cdot) \geq 0 \\ h_2(\cdot) \geq 0 \\ \vdots \\ h_N(\cdot) \geq 0 \end{array} \Rightarrow g(\cdot) \geq 0$$

$$\bigcap_{1 \leq i \leq N} \{x \in \mathbb{R}^n : h_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : g(x) \geq 0\}$$

$$\bigcap_{1 \leq i \leq N} \{x \in \mathbb{R}^n : h_i(x) \geq 0\} \cap \{x \in \mathbb{R}^n : g(x) < 0\} = \varnothing$$

If there exist $\{\lambda_i \geq 0\}_{i=1}^N$ such that

$$g(x) - \sum_{i=1}^N \lambda_i h_i(x) \geq 0 \quad \forall x \in \mathbb{R}^n$$

then containment/intersection condition holds.

If there exist $\{\lambda_i \geq 0\}_{i=1}^N$ such that

$$G - \sum_{i=1}^N \lambda_i H_i \succeq 0$$

then containment/intersection condition holds.

"S-procedure" on $\mathbb{R}\uparrow n$

Easy-to-apply for quadratic g, h

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T M \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0 \ \forall x \in \mathbb{R}^n \quad \Leftrightarrow \quad M \succeq 0$$

Include list of equalities too

Sufficient condition

Classic upper bounds: Doyle; Doyle/Fan/Tits

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

$$\mathcal{D} := \left\{ \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & d_3 I_{m_3} \end{bmatrix} : D_1 = D_1^* \succ 0, D_2 = D_2^* \succ 0, d_3 > 0 \right\}$$

$$\mathcal{G} := \left\{ \begin{bmatrix} G_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : G_1 = G_1^* \right\}$$

If $\beta > 0$, and $G \in \mathcal{G}, D \in \mathcal{D}$ satisfy

$$M^* D M - \beta^2 D + j(GM - M^* G) \preceq 0,$$

then $\mu_{\mathbf{\Delta}}(M) \leq \beta$.

Classic upper bounds: Doyle; Young/Newlin/Doyle

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

$$\mathcal{D} := \left\{ \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & d_3 I_{m_3} \end{bmatrix} : D_1 = D_1^* \succ 0, D_2 = D_2^* \succ 0, d_3 > 0 \right\}$$

$$\mathcal{G}_\sigma := \left\{ \begin{bmatrix} \mathrm{diag}[g_1, \ldots, g_{t_1}] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : g_k \in \mathbb{R} \right\}$$

If $\beta > 0$, and $G \in \mathcal{G}_\sigma, D \in \mathcal{D}$ satisfy

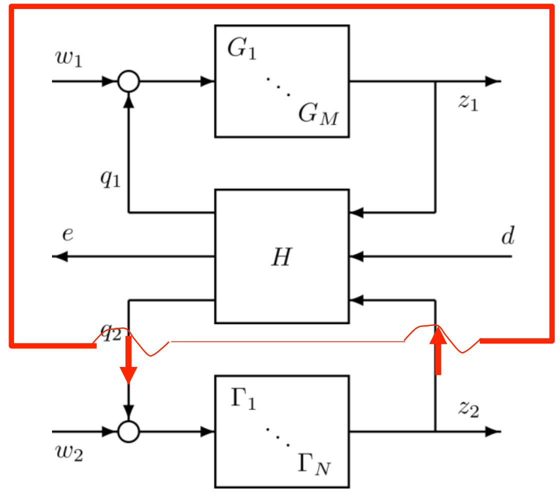
$$\bar{\sigma}\left[(I + G^2)^{-\frac{1}{4}} \left(\frac{1}{\beta} DMD^{-1} - jG\right)(I + G^2)^{-\frac{1}{4}}\right] \leq 1,$$

then $\mu_{\mathbf{\Delta}}(M) \leq \beta$.

Frequency-Grid: adaptive peak estimation

Theorem: (G, H, Γ) interconnection is well-posed and stable for all $\Gamma \in \mathbf{\Gamma}$ with $\|\Gamma\|_\infty < \frac{1}{\beta}$ if and only if

$$M \in \mathbb{R}H_\infty^{n\times n} \text{ and}$$
$$\max_{\omega \in \mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq \beta$$



$\mu_{\mathbf{\Delta}}(M(j\bar{\omega})) < \beta$ if $\beta > 0$, and $G \in \mathcal{G}_\sigma, D \in \mathcal{D}$ satisfy

$$\bar{\sigma}\left[(I+G^2)^{-\frac{1}{4}}\left(\frac{1}{\beta}DM(j\bar{\omega})D^{-1} - jG\right)(I+G^2)^{-\frac{1}{4}}\right] < 1.$$

Use Hamiltonian techniques to find interval,

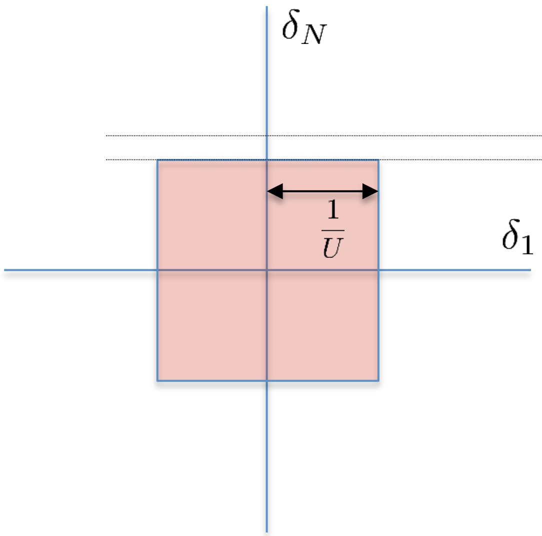
$$\omega_L < \bar{\omega} < \omega_R$$

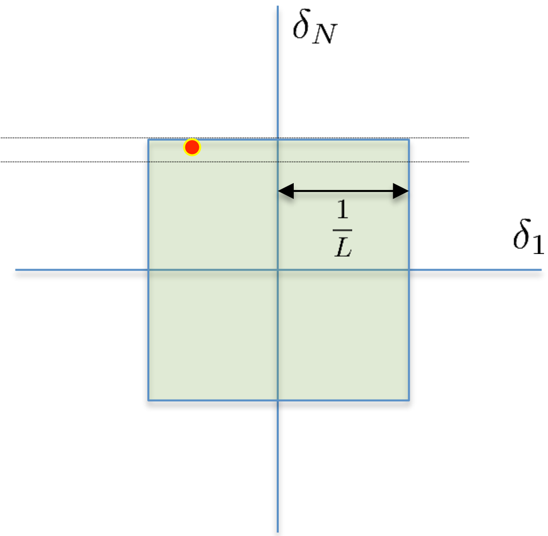
for which this D, G certify $\mu_{\mathbf{\Delta}}(M(j\omega)) \leq \beta$ for all $\omega \in [\omega_L, \omega_R]$

Upper and Lower Bounds

$\mu_{\boldsymbol{\Delta}}(M) < U$

$L \leq \mu_{\boldsymbol{\Delta}}(M)$





No roots in this box

root in this box

Lower Bounds: trying to find small roots of *det(I−MΔ)*

Optimality conditions at minimum-norm root:

if $\beta = \mu_{\mathbf{\Delta}}(M)$, then there exist $z, w, a, b \in \mathbb{C}$ and $q \in \mathbb{R}$ solving

$$Mb = \beta a$$

$$z_1 = q w_1, \quad z_2 = \frac{w_2^* a_2}{|w_2^* a_2|} w_2, \quad z_3 = \frac{\|w_3\|}{\|a_3\|} a_3$$

$$M^* z = \beta w$$

$$b_1 = q a_1, \quad b_2 = \frac{a_2^* w_2}{|a_2^* w_2|} a_2, \quad b_3 = \frac{\|a_3\|}{\|w_3\|} w_3$$

for $q \in [-1, 1]$ with

$$\begin{array}{ll} \mathrm{Re}(a_1^* w_1) \geq 0 & \text{if } q = 1 \\ \mathrm{Re}(a_1^* w_1) \leq 0 & \text{if } q = -1 \\ \mathrm{Re}(a_1^* w_1) = 0 & \text{if } |q| < 1 \end{array}$$

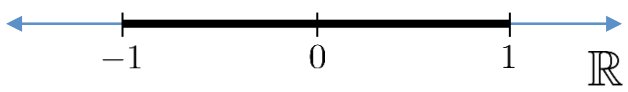
$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

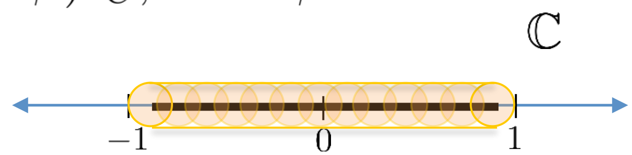
Conversely, if (β, z, w, a, b, q) solve these equations, then $\beta \leq \mu_{\mathbf{\Delta}}(M)$

Devise iteration, where fixed-point is a solution. Connections to common existing iterative algorithms in special cases. All equilibrium points give lower bound for $\mu_{\mathbf{\Delta}}(M)$, and produce offending $\Delta \in \mathbf{\Delta}$

Important regularizations

Replace real-parameters $\quad \delta_R \to \beta\delta_R + (1-\beta)\delta_C, \quad 1 \approx \beta < 1$





- guarantees continuity of $\mu_{\mathbf{\Delta}} : \mathbb{C}^{n\times n} \to \mathbb{R}$
- improves lower-bound convergence
- easily interpreted approximation of uncertain gain-like properties with slight dynamic characteristics

Replace time-invariant dynamics with arbitrarily-slowly time-varying dynamics

$$\Delta \in \mathbb{R}H_\infty, \|\Delta\|_\infty \leq 1 \qquad \to \qquad \Delta \text{ LTV} \qquad \|z\Delta - \Delta z\|_{2,2} < \epsilon \qquad \|\Delta\|_{2,2} < 1$$

- DMD^{-1} upper bound for μ is exact answer
- easily interpreted approximation of time-invariant uncertainty with arbitrarily-slowly, time-varying uncertainty

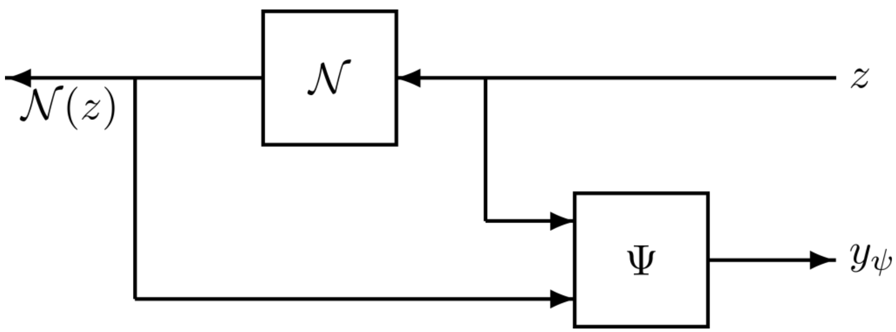
Moving beyond LTI uncertainty

Definition: Suppose Ψ is a stable linear system and M is a symmetric matrix. A bounded operator $\mathcal{N}$ satisfies the *hard IQC defined by* (Ψ, M) if

$$\int_0^T y_\psi^T(t) M y_\psi(t)dt \geq 0$$

Megretski and Rantzer, 1997, IEEE TAC, "System analysis via Integral Quadratic Constraints"

for all T and all signals $z \in \mathbf{L}_2^e[0\ \infty)$, with $y_\psi := \Psi \begin{bmatrix} z \\ \mathcal{N}(z) \end{bmatrix}$

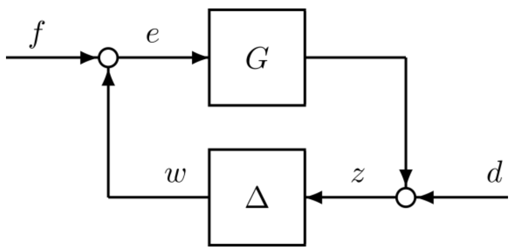


$z \in \mathbf{L}_2^e[0\ \infty)$

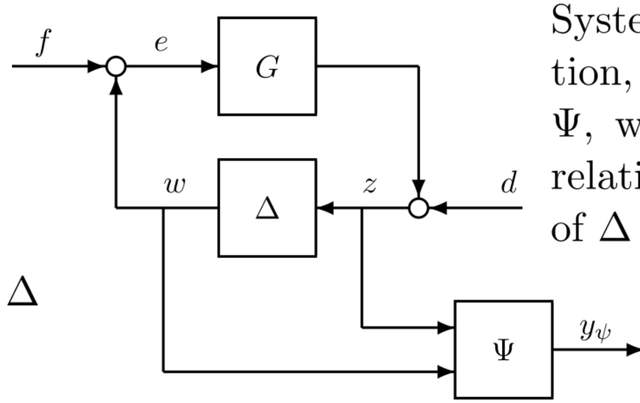
$$\int_0^T y_\psi^T(t) M y_\psi(t)dt \geq 0$$

$$\int_0^\infty y_\psi^T(t) M y_\psi(t)dt \geq 0$$

Three systems



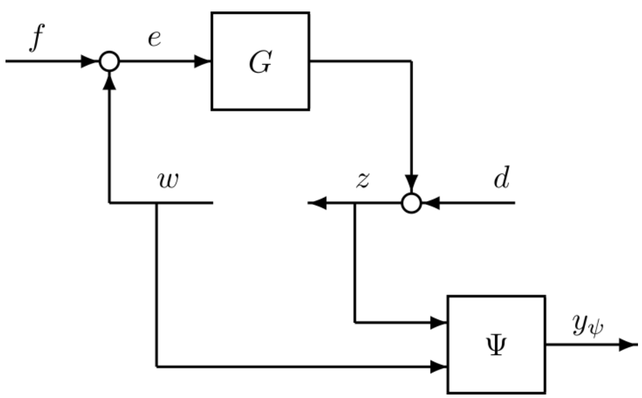
System under consideration, with "unknown" Δ



System under consideration, augmented with known Ψ, which captures "correlations" in input/output of Δ

… to reach conclusions here

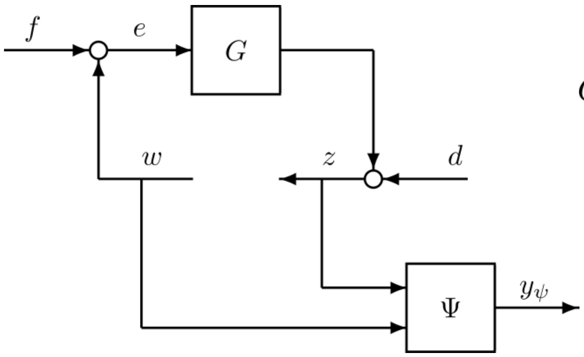
Analyze this… (system model and signal constraint)



System under consideration, with "unknown" Δ removed, but known augmented correllator Ψ (implicitly) providing information about signals

and $$\int_0^T y_\psi^T(t) M y_\psi(t)\,dt \geq 0$$

What are the known constraints?



$$G : \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad \Psi : \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

$$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0 \quad \forall T$$

Inequality constraint

- $\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0$
- extra information about z and w

Equality constraints

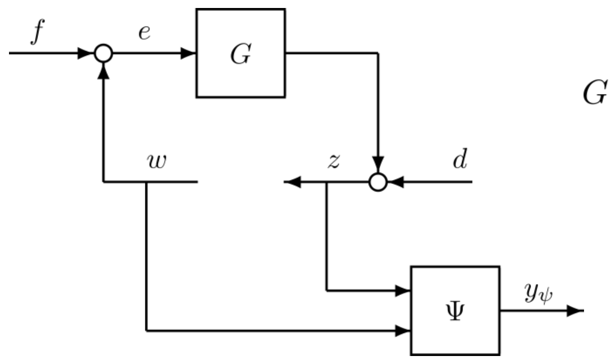
- Summing junctions, eg., $e = f + w$
- ODE models of G and Ψ

Under what conditions do these constraints actually imply a constraint between (f, d) and (e, z)? Specifically,

$$\int_0^T e^T(t)e(t) + z^T(t)z(t)dt \leq \gamma^2 \int_0^T f^T(t)f(t) + d^T(t)d(t)dt$$

Easy approach: Use Lyapunov-like construction and S-procedure...

Analyzing the constraints



$$G : \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad \Psi : \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

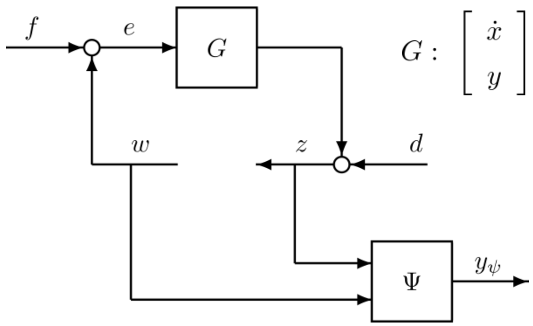
$$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0 \quad \forall T$$

Lyapunov + S-procedure: If there exists a positive, semidefinite function $V(x,\eta)$ and $\lambda \geq 0$ such that

$$\dot{V}(x,\eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2 (f^T f + d^T d) - (e^T e + z^T z)$$
$$\left(\dot{V} := \nabla_x V \cdot (Ax + Be) + \nabla_\eta V \cdot (\bar{A}\eta + \bar{B}_1 z + \bar{B}_2 w) \right)$$

for **all** values of x, η, d, f, w, e, z and y_ψ, constrained only by the interconnection, then the desired relation holds. **Why?** Integrate, from $x(0) = 0, \eta(0) = 0$, and use known *integral quadratic constraint* on y_ψ.

Analysis Inequality is an SDP



$$G: \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad \Psi: \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix} \qquad \int_0^T y_\psi^T(t) M y_\psi(t)\, dt \geq 0 \quad \forall T$$

$$\dot{V}(x,\eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2 (f^T f + d^T d) - (e^T e + z^T z)$$

Inequality in 5 variables (x, η, f, w, d)

$$s := \begin{bmatrix} x \\ \eta \\ f \\ d \\ w \end{bmatrix}$$

(e.g.) Restrict attention to <u>quadratic</u> $V(x,\eta) := \begin{bmatrix} x \\ \eta \end{bmatrix}^T P \begin{bmatrix} x \\ \eta \end{bmatrix}$ for some $P = P^T \succeq 0$.

Inequality becomes: $\quad s^T \left[\text{ Linear in } P, \lambda, \gamma^2 \ \right] s \leq 0 \quad \forall s \in \mathbf{R}^{n_x + n_\eta + n_f + n_d + n_w}$

IQC Analysis: Does there exist $P = P^T \succeq 0$, $\lambda \geq 0$, $\gamma_s > 0$ with

$$M(P, \lambda, \gamma_s) \preceq 0,$$

which is yet another (important) example of a semidefinite program.

Connections: Frequency and Time domain

Megretski and Rantzer make weaker assumptions about constraints the operator satisfies

– nonnegativity must hold only on

(but a slightly stronger well-posedness assumption).

Contrast to the Lyapunov argument, where for all is used explicitly

Moreover, the MR analysis condition (expressed in frequency-domain) is easier to satisfy

– Equivalent to dropping the semi-definiteness requirement on

Recent work (Seiler, to appear, TAC) shows the distinction can be less significant appears. Under mild technical assumptions, a PSD quadratic Lyapunov-based ce always exists whenever the MR frequency-domain condition holds.

Time-domain formulation is easily applied to LTV, LPV, finite-horizon, etc.

Higher-order S-procedure

$$\bigcap_{1\le i\le N} \{x \in \mathbb{R}^n : h_i(x) \ge 0\} \subseteq \{x \in \mathbb{R}^n : g(x) \ge 0\}$$

$$\bigcap_{1\le i\le N} \{x \in \mathbb{R}^n : h_i(x) \ge 0\} \cap \{x \in \mathbb{R}^n : g(x) < 0\} = \varnothing$$

If there exist $\{\lambda_i \ge 0\}_{i=1}^N$, and $\{\tau_{ij} \ge 0\}_{i,j}$ such that

$$g(x) - \sum_{i=1}^N \lambda_i h_i(x) - \sum_{i,j=1}^N \tau_{ij} h_i(x) h_j(x) \ge 0 \quad \forall x \in \mathbb{R}^n$$

then containment/intersection condition holds.

A higher-order "S-procedure" on $\mathbb{R}\uparrow n$

- quadratic g, h lead to quartic expression
  - need to check nonnegativity…

and so on …

Sum-of-squares (SOS) to verify nonnegativity

Is the polynomial

$$p(x) = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$$

non-negative, everywhere?

Yes – as it can be rearranged to

$$p(x) = \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1x_2)^2 \; + \; \frac{1}{2}(x_2^2 + 3x_1x_2)^2 \quad \in \mathrm{SOS}_x$$

Determining if rearrangement into a sum-of-squares is possible is theoretically easy for any polynomial

- Eigenvectors/eigenvalues for quadratics
- Practical and reliable for general quartics in 10s of variables
- Semidefinite program (SDP)
- Simplifications for polynomials with sparse representation in monomial basis

Parrilo, CDS, 2000+

SOS optimization: linear objective, SOS constraints

Data: collection of polynomials

$$\{f_{k,j}\}_{k=0,\ldots,M;j=1,\ldots,d}$$

$$\min_{z \in \mathbb{R}^M} c^T z$$

$$\begin{array}{c}
f_{0,1}(x) + z_1 f_{1,1}(x) + \cdots + z_M f_{M,1}(x) \in \mathrm{SOS}_x \\
f_{0,2}(x) + z_1 f_{1,2}(x) + \cdots + z_M f_{M,2}(x) \in \mathrm{SOS}_x \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
f_{0,d}(x) + z_1 f_{1,d}(x) + \cdots + z_M f_{M,d}(x) \in \mathrm{SOS}_x
\end{array}$$

Solved with semidefinite programming

Implementing a higher-order S-procedure

$$\bigcap_{1\le i\le N}\{x\in\mathbb{R}^n : h_i(x)\ge 0\}\subseteq\{x\in\mathbb{R}^n : g(x)\ge 0\}$$

$$\bigcap_{1\le i\le N}\{x\in\mathbb{R}^n : h_i(x)\ge 0\}\cap\{x\in\mathbb{R}^n : g(x)< 0\}=\varnothing$$

If there exist $\{\lambda_i\in \mathrm{SOS}_x\}_{i=1}^N$, and $\{\tau_{ij}\ge 0\}_{i,j}$ such that

$$g(x)-\sum_{i=1}^N\lambda_i(x)h_i(x)-\sum_{i,j=1}^N\tau_{ij}h_i(x)h_j(x)\in\mathrm{SOS}_x$$

then containment/intersection condition holds.

Use SOS-optimization

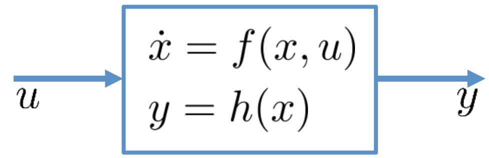
– choose basis to linearly parametrize the λ

$$\lambda_i(x)=\sum_{k=1}^{n_b}\gamma_{ik}\phi_k(x)$$

– SOS decision variables are $\{\gamma_{ik},\tau_{ij}\}$

SOS to verify Input/Output properties

$\mathcal{L}_2$ gain



$$\nabla V \cdot f(x,u) \leq \gamma^2 u^T u - h(x)^T h(x) \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^{n_u}$$

Dissipative with respect to supply-rate $s(\cdot,\cdot)$

$$\nabla V \cdot f(x,u) \leq s(u, h(x)) \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^{n_u}$$

Restrict to polynomial f, h, V, s

Satisfies IQC defined by $\Psi := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $M = M^T$.

use SOS

Decision variables:

nonlinear Ψ

$$\nabla_x V \cdot f(x,u) + \nabla_\eta V \cdot \left(A\eta + B\begin{bmatrix} u \\ h(x) \end{bmatrix}\right) \leq$$
$$\left(C\eta + D\begin{bmatrix} u \\ h(x) \end{bmatrix}\right)^T Q \left(C\eta + D\begin{bmatrix} u \\ h(x) \end{bmatrix}\right)^T \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_u}$$

Input/Output properties (local)

(e.g.) Locally-dissipative with respect to supply-rate $s(\cdot, \cdot)$

$$\nabla V \cdot f(x, u) \leq s(u, h(x)) \quad \text{on} \quad \{x : V(x) \leq R\} \times \mathcal{U}$$

SOS analysis (for example), maximizing a parameter ρ in $s = s_0 + \rho s_1$

$$s(u, h(x)) - \nabla V \cdot f(x, u) - \lambda_1(x, u)(R - V(x)) - \lambda_2(x, u) g_{\mathcal{U}}(u) \in \text{SOS}_{x,u}$$

Difficulty: decision variables λ_1 and V enter bilinearly. Approach: Iterate

- Hold V fixed, optimize ρ over λ_1, λ_2
- Hold $\rho, \lambda_1, \lambda_2$ fixed, recenter (analytic center) V

Region-of-attraction

Dynamics, equilibrium point

$$\dot{x}(t) = f(x(t)), \qquad f(\bar{x}) = 0$$

p: Analyst-defined function whose (well-understood) sub-level sets are to be in region-of-attraction

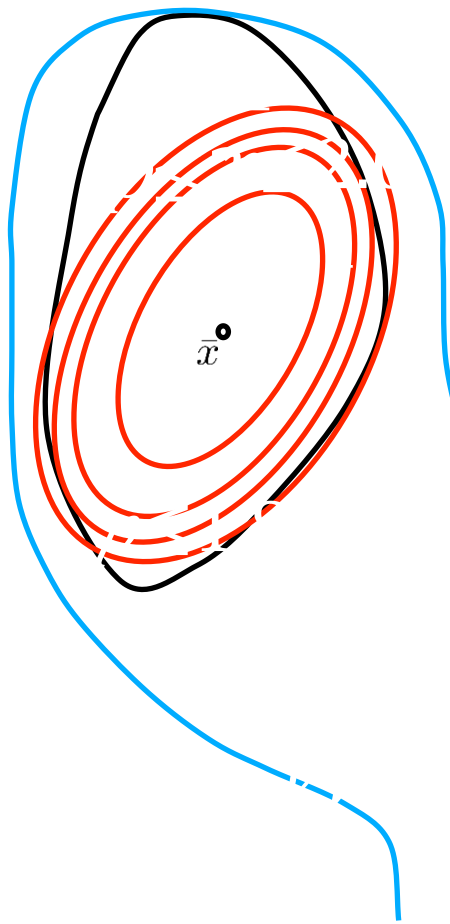
$$\{x : p(x) \leq \beta\} \subseteq \mathrm{ROA}_{\bar{x}}$$

By choice of positive-definite *V*, maximize *β* so that:

$$\{x : p(x) \leq \beta\} \subseteq \{x : V(x) \leq 1\}$$

$$\{x : V(x) \leq 1\} \quad \text{is bounded}$$

$$\{x \neq \bar{x} : V(x) \leq 1\} \subseteq \{x : \nabla V \cdot f(x) < 0\}$$

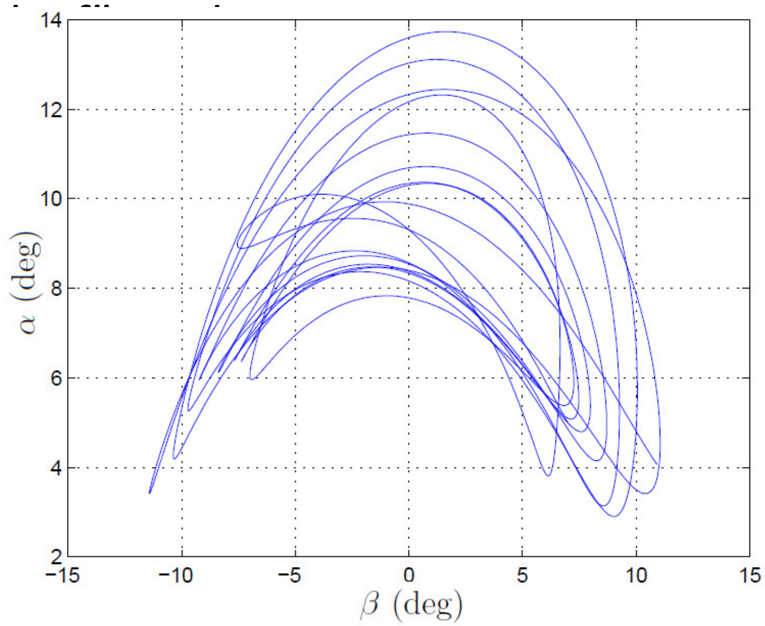


- Certify containments with S-procedure
- Use SOS to decide (≥ 0)
- Decision variables: $V, \lambda,$

- Bilinear: use iteration
  - Optimize in some steps
  - Center in others

F-18 Falling-Leaf mode

The US Navy has lost many F/A-18 A/B/C/D Hornet aircraft due to an out-of-control flight departure phenomenon described as the “falling



Can require 15,000-20,000 ft to recover

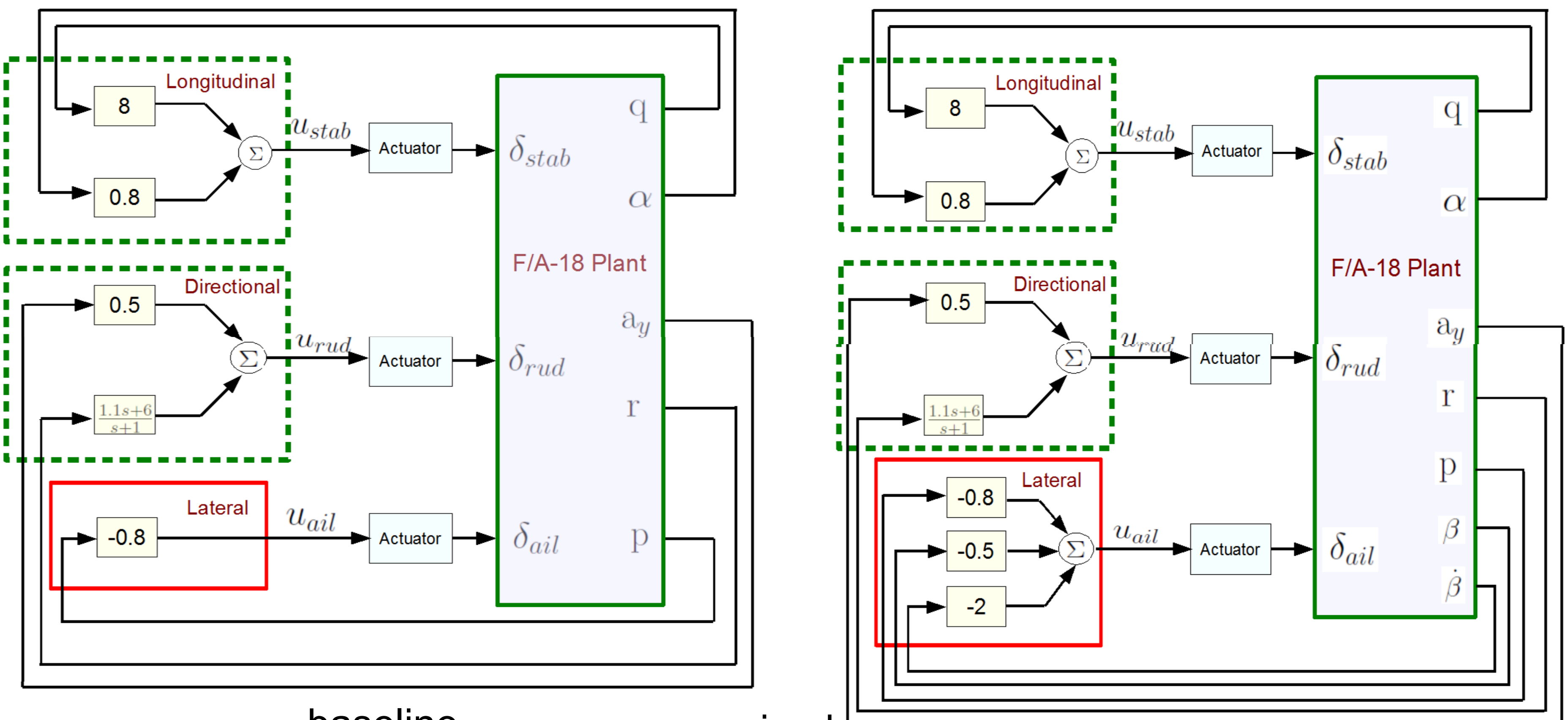
Administrative action by NAVAIR to prevent further losses

Revised control law implemented, deployed in 2003/4, F/A-18E/F

– uses ailerons to damp sideslip

Heller, David and Holmberg, “Falling Leaf Motion Suppression in the F/A-18 Hornet with Revised Flight Control Software," AIAA-2004-542, 42nd AIAA Aerospace Sciences Meeting, Jan 2004, Reno, NV.

Simplified FCS



baseline

revised

Linearized analysis cannot discriminate here

Linearized Analysis: at equilibrium and several steady turn rates

- Classical loop-at-a-time margins
- Disk margin analysis (Nichols)
- Multivariable input disk-margin
- Diagonal input multiplicative uncertainty
- "Full"-block input multiplicative uncertainty
- μ-analysis using physically motivated uncertainty in 8 aero coefficients

<u>Conclusion: Both designs have excellent (and nearly identical) linearized robustness margins trimmed across envelope…</u>

ROA analysis

Perform region-of-attraction estimate as described ... r) for direct

$$\dot{\beta} = 0.20127\alpha^2\beta - 0.0015591\alpha^2\mathrm{p} - 0.0021718\alpha^2\mathrm{r} + 0.0019743\alpha^2 x_c + 0.32034\alpha\beta\mathrm{q} + 0.065962\beta^3$$
$$+ 0.17968\alpha\beta + 0.98314\alpha\mathrm{p} - 0.023426\alpha\mathrm{r} - 0.024926\alpha x_c + 0.134\beta\mathrm{q} + 0.0025822\alpha - 0.0068553\beta$$
$$+ 0.45003\mathrm{p} + 0.1288\phi - 0.99443\mathrm{r} + 0.0056922 x_c$$
$$\dot{\mathrm{p}} = 17.7160\alpha^2\beta - 0.0277\alpha^2\mathrm{p} - 0.0386\alpha^2\mathrm{r} + 0.0351\alpha^2 x_c - 0.0033\beta^3 + 2.1835\alpha\beta + 3.0420\alpha\mathrm{p}$$
$$- 0.4139\alpha\mathrm{r} - 0.4 \ldots$$
$$+ 0.2723 x_c$$
$$\dot{\mathrm{r}} = -1.4509\alpha^2\beta + 0. \ldots$$
$$+ 0.1410\alpha\mathrm{r} + 0.1 \ldots$$
$$\dot{\phi} = \mathrm{p}$$
$$\dot{\alpha} = -\alpha\beta\mathrm{r} + 0.2467\alpha \ldots$$
$$\dot{\mathrm{q}} = 0.5196\alpha^2 + 4.861 \ldots$$
$$\dot{x_c} = 4.9\mathrm{r} - x_c$$

M...

$$\dot{\beta} = 0.1831\alpha^2\beta - 0.0496\alpha^2\mathrm{p} - 0.0005\alpha^2\phi + 0.0017\alpha^2\mathrm{r} + 0.0030\alpha^2 x_c + 0.3203\alpha\beta\mathrm{q} + 0.0643\beta^3$$
$$+ 0.0027\alpha\beta + 0.9557\alpha\mathrm{p} - 0.0054\alpha\phi + 0.0187\alpha\mathrm{r} - 0.0250\alpha x_c + 0.1340\beta\mathrm{q} + 0.0026\alpha - 0.0091\beta$$
$$+ 0.4457\mathrm{p} + 0.1276\phi - 0.9850\mathrm{r} + 0.0056 x_c$$
$$\dot{\mathrm{p}} = 1.1530\alpha^2\beta + 6.6577\alpha^2\mathrm{p} - 0.0082\alpha^2\phi + 0.0308\alpha^2\mathrm{r} - 0.1205\alpha^2 x_c + 18.3689\beta^3 - 0.5080\alpha\beta$$
$$+ 2.4908\alpha\mathrm{p} + 0.8743\alpha\phi - 7.2037\alpha\mathrm{r} - 0.3495\alpha x_c - 0.8151\mathrm{qr} - 0.0109\alpha - 4.6009\beta$$
$$- 3.5186\mathrm{p} - 0.4703\phi - 0.1096\mathrm{q} + 3.9316\mathrm{r} + 0.2527 x_c$$
$$\dot{\mathrm{r}} = -1.4275\alpha^2\beta + 0.0546\alpha^2\mathrm{p} + 0.0031\alpha^2\phi - 0.0117\alpha^2\mathrm{r} - 0.0132\alpha^2 x_c + 0.0079\beta^3 - 1.0008\alpha\beta$$
$$- 0.0096\alpha\mathrm{p} - 0.0029\alpha\phi + 0.1638\alpha\mathrm{r} + 0.1832\alpha x_c - 0.7544\mathrm{pq} - 0.0182\alpha + 0.1854\beta + 0.0895\mathrm{p}$$
$$+ 0.0124\phi - 0.3509\mathrm{r} - 0.1539 x_c$$
$$\dot{\phi} = \mathrm{p}$$
$$\dot{\alpha} = -\alpha\beta\mathrm{r} + 0.2467\alpha^2 - 0.1344\alpha\beta + 0.1473\alpha\mathrm{q} - \beta\mathrm{p} - 0.4538\beta\mathrm{r} - 0.2487\alpha - 0.0609\beta + 0.7139\mathrm{q}$$
$$\dot{\mathrm{q}} = 0.5196\alpha^2 + 4.8613\alpha\mathrm{q} + 0.97126\mathrm{pr} - 1.9162\alpha - 6.8140\mathrm{q} + 0.1305\mathrm{p}$$
$$\dot{x_c} = 4.9\mathrm{r} - x_c$$

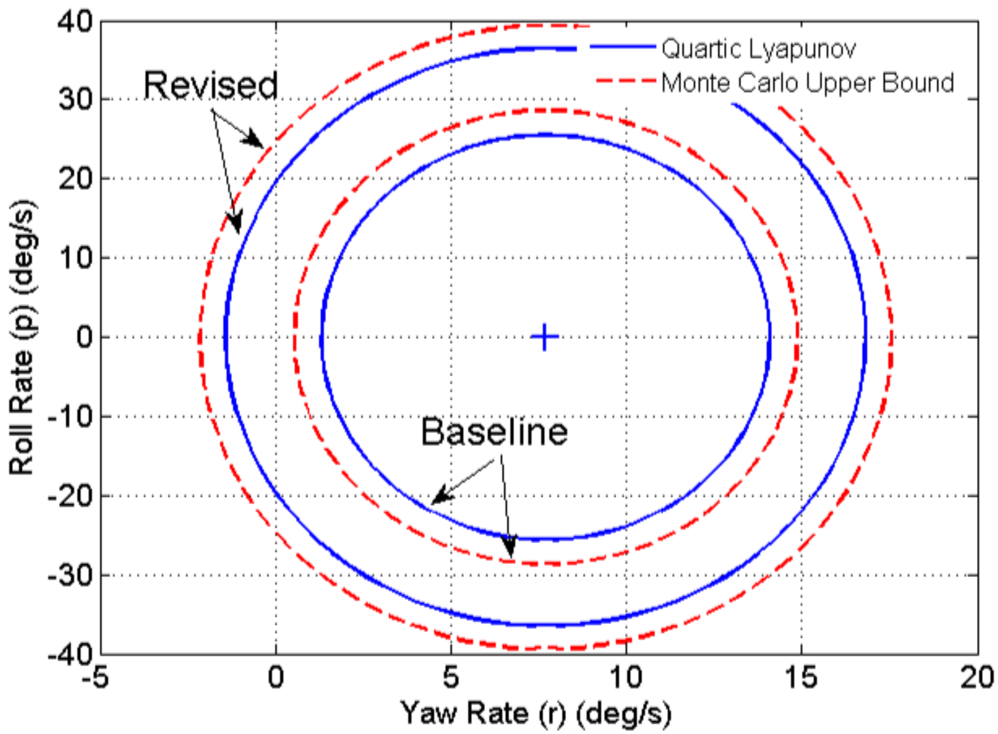
$$\begin{bmatrix} \alpha \\ q \\ x_c \end{bmatrix}$$

– polynomial

ROA analysis does discriminate

Ellipsoidal shape factor, aligned w/ states, appropriated scaled

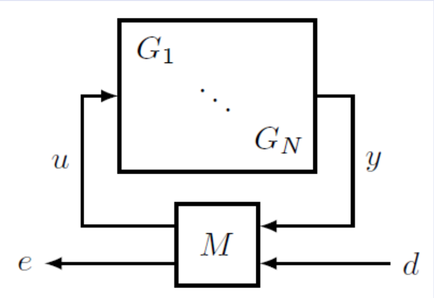
- O(1) hours for quartic Lyapunov function certificate
- O(100) hours for divergent sims with “small” initial conditions



Analysis of large-scale interconnection

Preliminary observations:

- Can analyze $\mathcal{L}_2$ gain, and more generally, dissipation properties of nonlinear systems with SOS (sum-of-squares) tools, searching for polynomial storage (Lyapunov) function.
- Small-scale SOS is possible ($\partial(f)=3$, $\partial(V)=4$, $n=6$)
- Large-scale SOS analysis appears difficult (dimension/scalability)



Proposed approach

1. Decompose large-scale nonlinear system to interconnection of small systems
2. Analyze each system's I/O properties, in light of interconnection
3. Combine properties/interconnection to make conclusion

Experience thus far

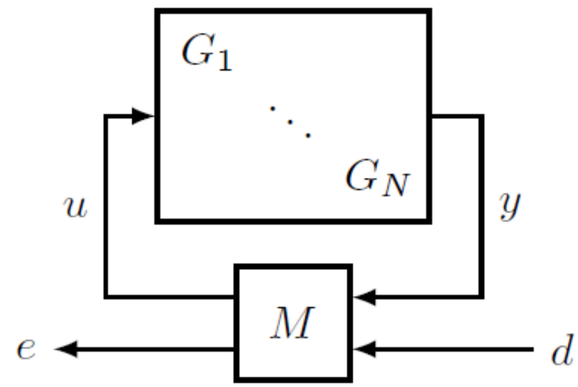
- If decomposition is already available, combining steps #2/#3 is not trivial, but…
- ADMM (Alternating Direction Method of Multipliers) is an optimization approach where #2 and #3 naturally fit
- Iteration is: a **negotiation** between a centralized analyst who is only aware of subsystem input/output properties; and individual agents who are aware of each subsystem's internal state model in order to confirm various input/output properties.

Result

- Framework for the certification of stability and input-output performance of interconnected dynamical systems
- Application to intended use-case has been challenging (eg., obtaining a tractable decomposition of F18 model)
- For systems which arise **naturally** as interconnection of a large number of smaller subsystems (power systems,

Distributed Analysis of interconnection

Search over arbitrary dissipativity properties satisfied by the subsystems, G_i in such a way that verification of the desired dissipativity property for the interconnected system (d,e) emerges.



dissipativity of i'th subsystem

dissipativity of interconnection

Feasibility of system analysis as optimization

$$\min_{x \in \mathbb{R}^n,\, z \in \mathbb{R}^m} \quad d(\ldots$$
$$\text{subject to:} \quad Ax + Bz = c$$

$$x = (V_1, Q_1; V_2, Q_2; \ldots; V_N, Q\ldots$$
$$d(x) = d_1(x_1) + d_2(x_2) + \cdots + d_N(x_N)$$
$$d_i(x_i) = \begin{cases} 0, & \text{if } \nabla V_i \cdot f_i(x_i, \ldots u_i) \le [\;]^T Q_i \begin{bmatrix} u_i \\ h_i(x_i, \ldots \end{bmatrix} \\ \ldots & \ldots\text{ise} \end{cases}$$

Indicator functions for dissipativity of individual subsystems

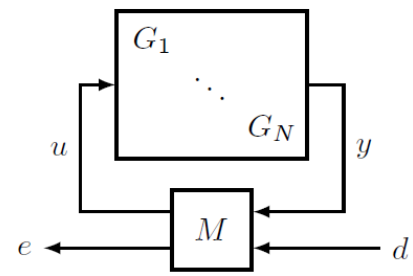
Subsystem supply rates in each analysis must match

Indicator function for dissipativity of interconnected subsystems, using dissipativity of subsystems

$$z = (Q_1, Q_2, \ldots, Q_N)$$
$$g(z) = \begin{cases} 0, & \text{if } \begin{bmatrix} M \\ I \end{bmatrix}^T L(Q_0, Q\ldots Q_2, \ldots, Q_N) \begin{bmatrix} M \\ I \end{bmatrix} \le 0 \\ \infty, & \text{otherwise} \end{cases}$$

Alternating direction method of multipliers

General ADMM



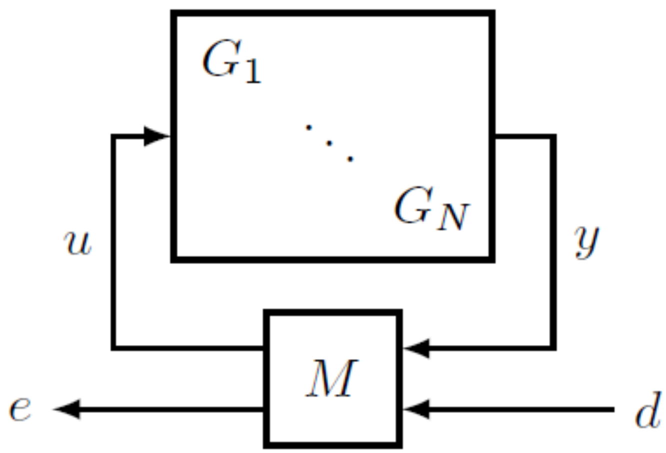
– Decoupled problems for subsystems: decision variables (V_i, Q_i)
  - Each system G_i independently establishes its dissipativity to a supply-rate Q_i that is as close as possible to a target supply-rate, Q_i (with a bias from the PI-action)
– Coupled problem for interconnection: decision variables $(Q_1, Q_2, \ldots, Q_N)$
  - Global input/output calculation to prove a system property on (d,e), via supply-rates $(Q_1, Q_2, \ldots, Q_N)$ that are as close as possible to individually certified for each subsystem, $(Q_1, Q_2, \cdots, Q_N)$, again with a bias from the PI-action
– PI control-action to force convergence of all $Q_i - Q_i \to 0$

Step-by-step interpretation

Nonlinear example

100 subsystems

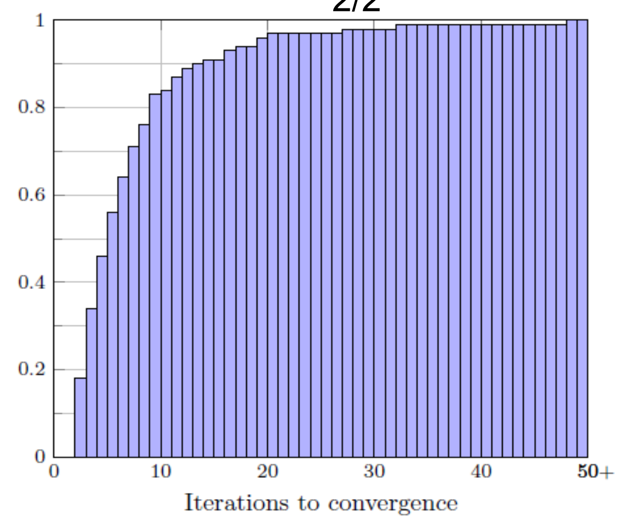
- 1-input, 1-output, 2 states
- Nonlinear, rational dynamics
- ∂=4 subsystem storage fcns in decoupled problems



Dense interconnection matrix M

Bounded $\mathcal{L}\downarrow 2$ gain, by construction:

2/2 1



Cumulative plot displaying the fraction of 200 total tests that required at most a given number of iterations to certify the L_2-gain property of the interconnected system. The fastest trials succeeded in 3 iterations and 90% succeeded in fewer than 15 iterations.

Nonlinear system construction

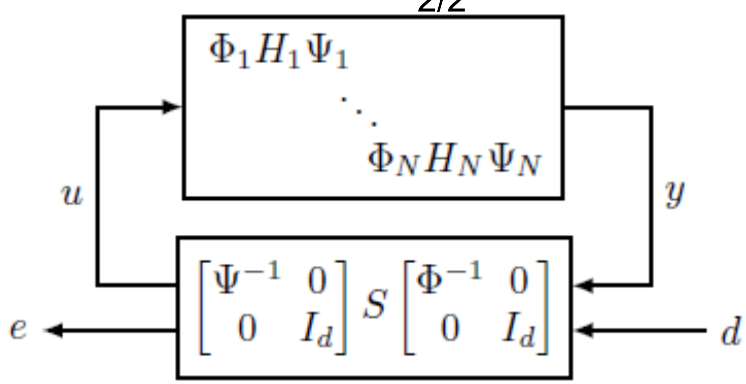
Each subsystem, Hi, is of the form:

$x1=x2$
$x2=-ax2-bx13-\ldots1+cx22$

$y=x2$

This has

2/2



Construction of the interconnected system:

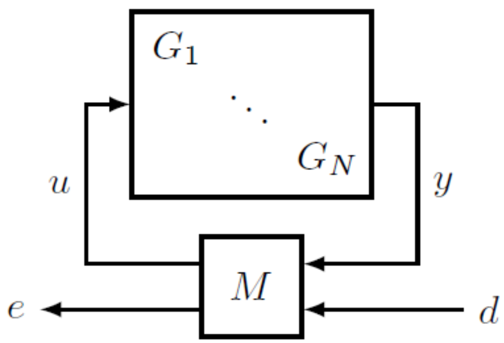
1. For the i-th subsystem choose $\{ai,b\downarrow i, c\downarrow i\}$ uniformly from $(1,2)\times(0,1)\times(0.5,2)$. Denote $\gamma := max i\; ai\uparrow -1$.
2. Choose $S\in R\uparrow(N+d)\times(N+d)$ from a normal distribution.
3. Compute $\beta := inf \beta(BMB\uparrow -1)$ where $B=diag(b1, ..., b\downarrow N, I\downarrow d), b\downarrow i \geq 0$. Redefine $S := 0.99/\gamma\beta S$. $M:=[\Psi\uparrow -1\; 0@0\&I\downarrow d$
4. Choose random nonzero, diagonal scalings $\Psi = diag(\Psi 1 \ldots \Psi N)$ and

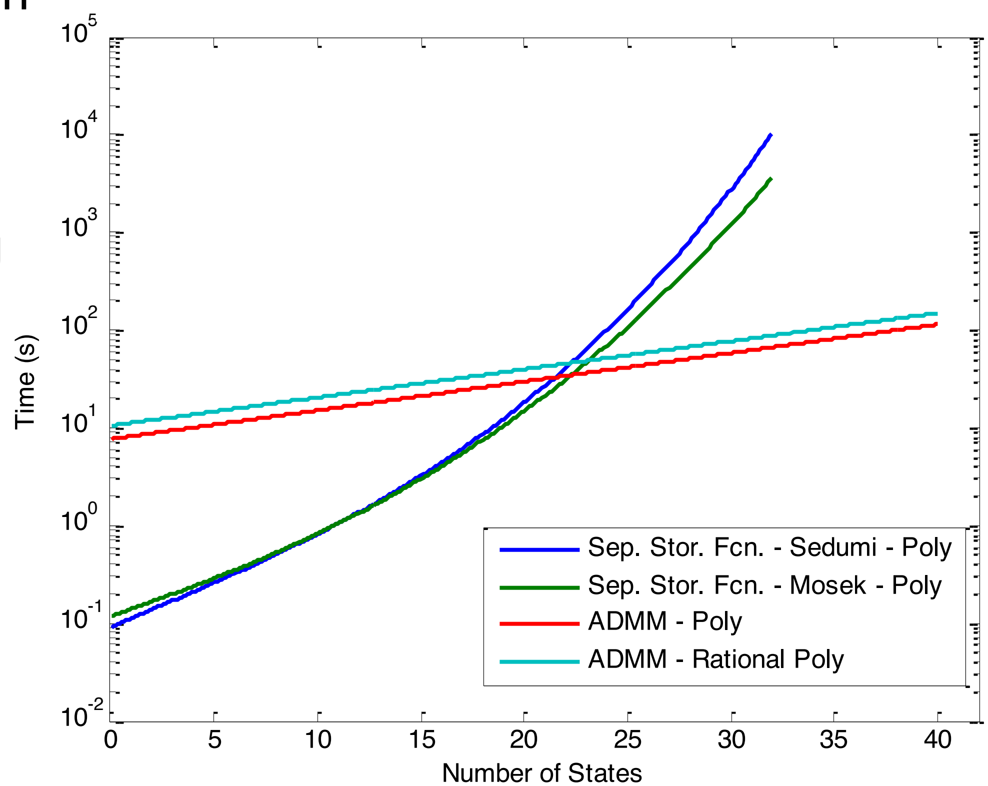
Comparing with direct additively-separable storage function

Decoupled analysis implicitly uses an additively-separable storage fcn

$$V(x) = V_1(x_1) + \cdots + V_N(x_N)$$

Can this be found directly, exploiting the complexity reduction (Newton polytope) in the SOS analysis?





IQCs

Instead of limited quadratic supply rates for dissipativeness, search over parametrized IQCs satisfied by the subsystems

Subsystem certification:

If is the realization of a stable linear system and

there exists a positive semidefinite such that

then , satisfies the IQC defined by

Interconnection certification:

$[\begin{smallmatrix} M \\ I \end{smallmatrix}]^T L(\Pi_0, \Pi_1, \ldots, \Pi_N)$

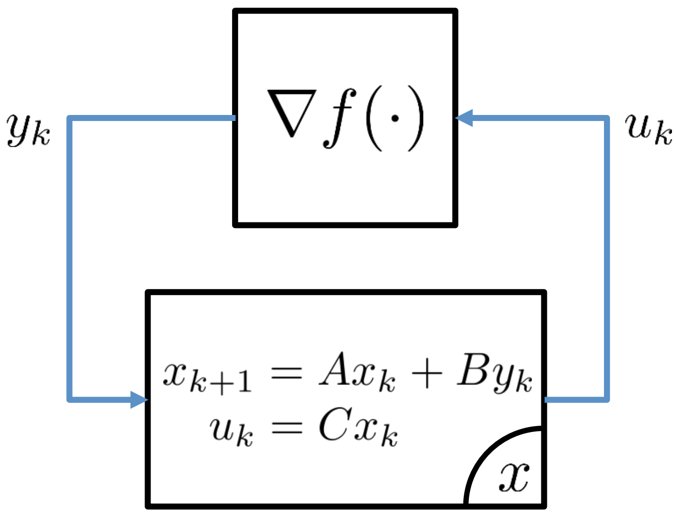
where $\Pi_i(\omega) = \Psi_i(\omega)^* Q_i \Psi$

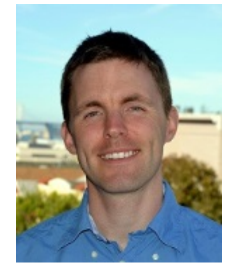
Interconnection constraint must hold for all frequencies. Tractable by sampling finitely many points or via the KYP lemma.

for all

First-order optimization algorithms as robust control

$$\min_{x \in \mathbf{R}^n} f(x)$$





Assumptions on f (uncertain plant)

- Strongly convex (m)
- Lipschitz gradients (L)

Algorithm (controller)

- Finite-dim, strictly proper, linear sys
- input: gradient at iterate
- output: next iterate

Automated Analysis with IQC/SDP

- characterize f with IQCs
- certify convergence-rate of interconnection

Extensions

- Gradient noise
- Constrained optimization
- Algorithm design

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1987 IFAC, Munich: Roy, Gary and John and Manfred

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Murat Arcak

Pete Seiler and Gary

Backups

Alternating direction method of multipliers

Optimization problem addressed

Usefulness depends on the structure of (d, g, A, B), and ease of solving the decoupled problems

ADMM algorithm

For convex f and g, mild conditions guarantee convergence of ADMM.

Alternating direction method of multipliers

General ADMM

Translation

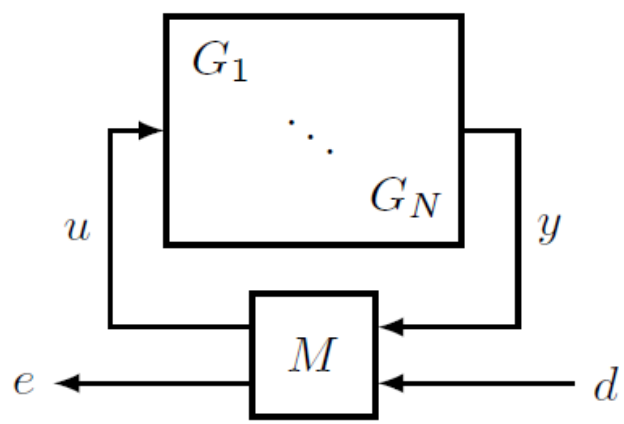
- Decoupled problems for subsystems, involving (V_i, Q_i)
- Coupled problem for interconnection, involving $(Q_1, Q_2, \ldots, Q_k)$
- Update to force convergence of $Q_i - Q_i \to 0$

1000 Linear example cases

50 subsystems

- 2-input, 2-output, 3 states

2×2 performance (d,e)

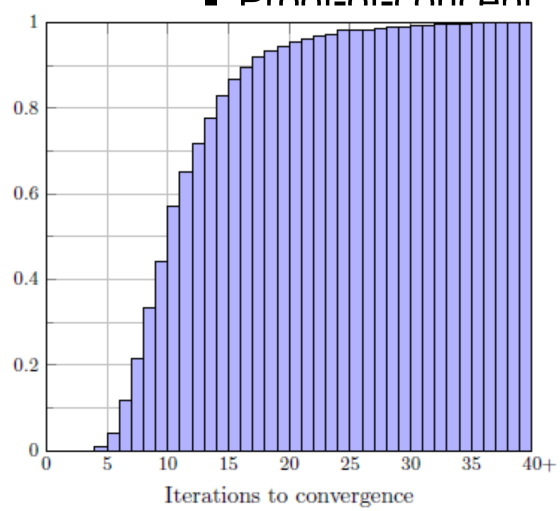


Sparse interconnection matrix M

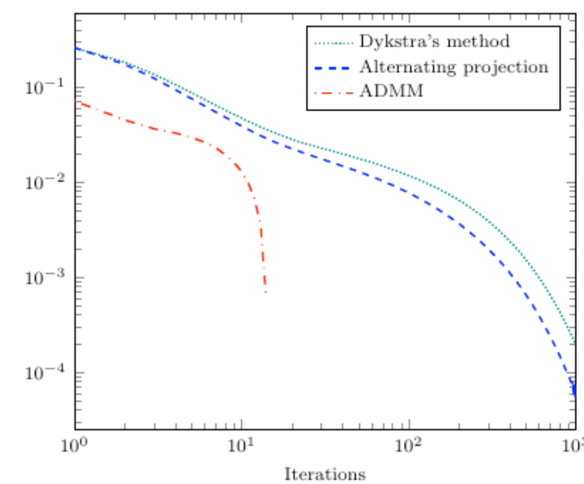
Bounded L_2 gain, by construction:

2/2 1

- Proof-of-concept



Cumulative plot showing the fraction of 1000 total trials that required at most a given number of iterations to find a feasible point using ADMM. For example, the fastest trials found a feasible point in 4 iterations. Also, 90% of trials succeeded in 16 iterations or fewer.



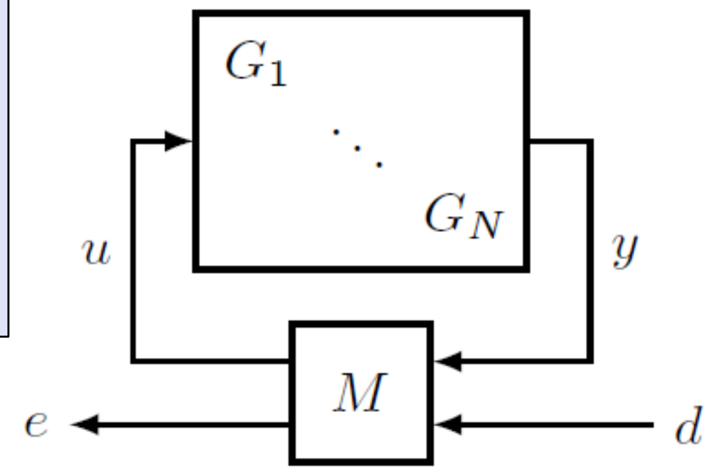
Plot of the largest eigenvalue for five different iterative methods. Feasibility is achieved when all eigenvalues are negative (indicated by a terminal circle). ADMM converged in 15 iterations, while the other methods took longer or failed to converge after 1000 iterations.

Unknown interconnection equilibrium

Allow uncertainty in subsystem dynamics so $(x_i = 0,\ u_i = 0, y_i = 0)$ may no longer be on the equilibrium manifold.

If equilibrium-manifolds of subsystems are not known, equilibrium-point of interconnection (even at

is not known.



Analyze dissipative properties of subsystem that are <u>relative to</u>, but <u>independent of</u> the equilibrium point (EID, for *equilibrium-independent dissipativity*)

Equilibrium-Independent Dissipativity

System dynamics

$$\dot{x}(t)=f(x(t),u(t)) \\ y(t)=h(x(t),u(t))$$

Equilibrium assumption: for every $u^* \in R^m$, there exists unique $x^* \in R^n$ such that $f(x^*, u^*)=0_n$

Call this the *equilibrium state-input map*

Define associated *equilibrium state-output map* $k_y: R^m \to R^q$, $k_y(u^*) := h(k_u(u^*), u^*)$, and $y^* := k_y(u^*)$

System is EID with respect to supply-rate w if for every $u^* \in R^m$, there exists non-negative storage function V_{u^*} with $\nabla V_{u^*}^T f(x,u) \le w(u-u^*, \ldots)$

for all ... and ...

Under this condition, along all solutions, and for every u^* and ..., with

$$V_{u^*}(x(T)) \le V_{u^*}(x(0)) + \int_0^T w(u(t)-u^*, y(t)-\ldots)\,dt$$

for all ...

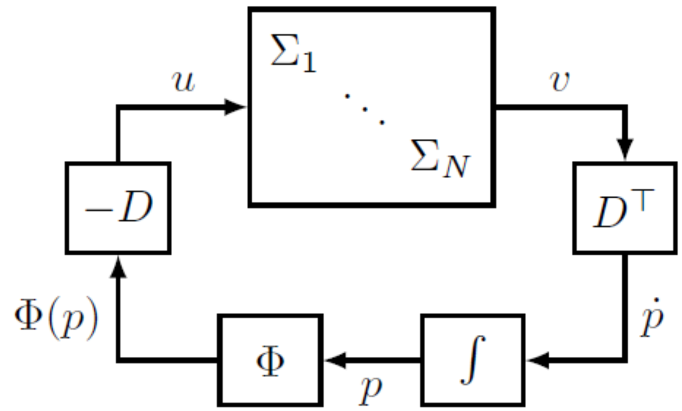
EID example of vehicle platooning

Consider a platoon of N vehicles:

- Each vehicle, Σi, measures its distance to other vehicles and adjusts its throttle according to a control law.
- A bidirectional graph with N nodes and L links defines the measurement topology.
- $D^T \in R^{L \times N}$ maps the vehicle velocities, v_i, to the relative velocities across each link, p_ℓ.
- We analyze control laws of the form

  where is surjective and

  increasing, but otherwise unknown.
- This guarantees the existence of an equilibrium point for the interconnected system, but it is unknown. Therefore, EID is used.



Each vehicle, Σi, is modeled as:

$$v\ i = -v_i + v_i^{n}$$
$$y_i = v_i$$

where is the

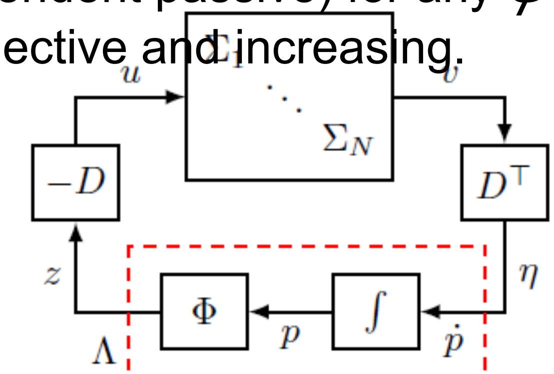
nominal velocity of the -th vehicle.

EID example of vehicle platooning: 20 vehicles

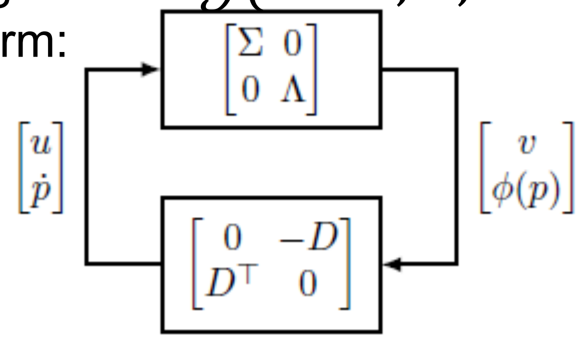
The map from p to $\Phi(p)$, indicated by the red dashed box, can be written as $\Lambda = diag(\Lambda_1, ..., \Lambda_L)$ where

$$\Lambda_\ell : \begin{cases} \dot{p}_\ell = \eta_\ell \\ z_\ell = \phi_\ell \end{cases}$$

Each Λ_ℓ is EIP (equilibrium independent passive) for any ϕ_ℓ that is surjective and increasing.



Letting $\Sigma = diag(\Sigma_1, ..., \Sigma_L)$ we can form:



The ADMM algorithm was applied with

- SOS programs to certify the EID of the subsystems, and
- the supply rates for each are fixed at

Summary, Extensions and Conclusions

Strategy:

Search over arbitrary dissipativity properties satisfied by the subsystems, G_j in such a way that the desired dissipativity property for the interconnected system emerges.

Extensions to make this relevant to large-scale nonlinear system certification

- Allow M to be a known linear system, not just a gain
- Use parametrized IQCs, not parametrized supply rates
- Certify subsystem dissipativity/IQCs using numerical schemes, not SOS-proofs

Conclusions:

This strategy is suitable for implementation using the ADMM approach.

The algorithm has some opportunities for trivial parallelization

Proof-of-concept on large linear and nonlinear systems

Can employ equilibrium-independent properties

Pragmatic transition plan to include analysis techniques for a wider variety of subsystem models

SOS optimization

Given $\{p_k\}_{k=0,1,\cdots,N}$, does there exist $\{\alpha_k\}_{k=1,\cdots,N}$ such that

Determining if a combination exists is theoretically easy

- SDP
- Practical, reliable for general quadratics in 100s variables; quartics in 10s of variables
- Simplifications for polynomials with sparse monomials

Building block to outer-bounds of minimization (or maximization)

subject to:

Hierarchy of outer-bounds to non-convex minimization

$$\max_{\beta,\lambda_k} \beta$$

$$q_0(x)-\beta-\sum_k \lambda_k q_k$$

$$\lambda_k \geq 0$$

Lower bound to minimum

subject to:

$$\max_{\beta,\lambda_k,\tau_{ij}} \beta$$

$$q_0(x)-\beta-\sum_k \lambda_k(x) q_k(x)-\sum_{i,j}$$

$$\lambda_k(x) \in SOS$$

$$\tau_{ij} \geq 0$$

Better lower bound to minimum