Computation and Uncertainty
The Past, Present and Future of Control

Manfred Morari

with thanks to
Paul Goulart, Alex Domahidi
and many other collaborators

Automatic Control Laboratory, ETH Zürich

Outline

- Past
  - Where we came from: A pre-history of CDS
- Present
  - Where we are: Fast MPC
- Future
  - Where we should be going: Open research areas

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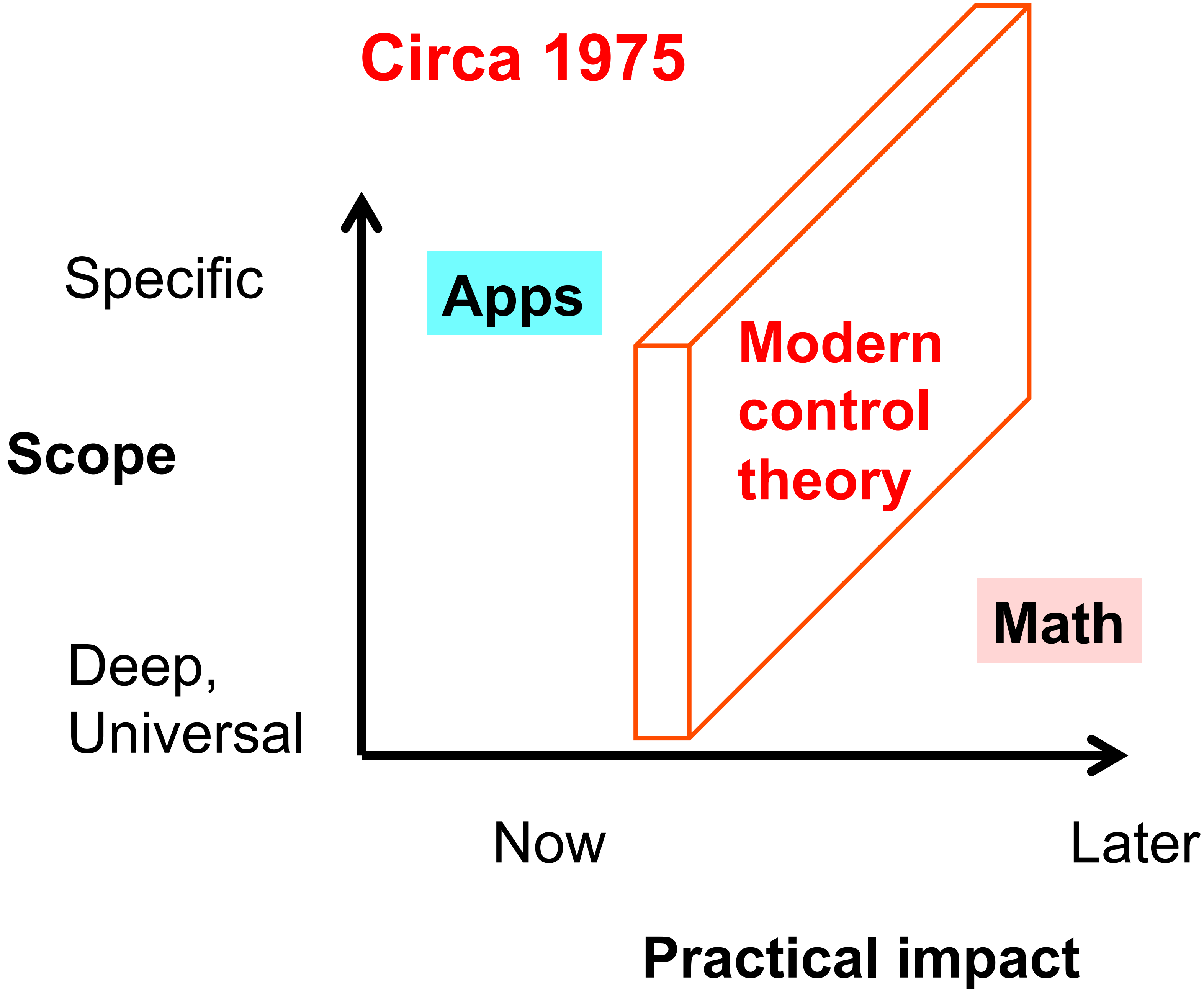


Circa 1975
Specific
Apps
Modern control theory
Scope
Math
Deep, Universal
Now
Later
Practical impact

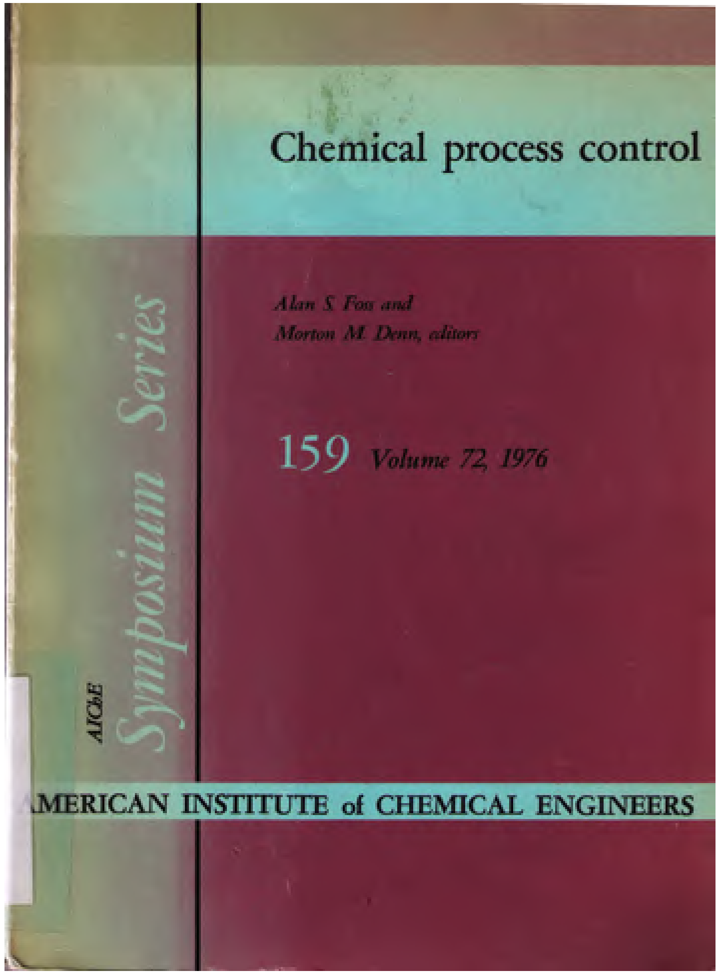
Theory-Practice Gap

Main theme of CPC I in 1976

Explosive development of theory had taken place

- Industry did not understand theory
- Academia had no clue about real controller design

Exceptions: Åström, Gilles, Balchen,…



Theory-Practice Gap: Model Uncertainty

- Control Objective did not address robustness / uncertainty directly. Indirect effect of tuning parameters was not understood (Horowitz, Shinnar, Doyle,…)

8 Ind. Eng. Chem. Process Des. Dev., Vol. 18, No. 1, 1979

Design of Sampled Data Controllers

Zalman J. Palmor[1] and Reuel Shinnar*

Department of Chemical Engineering, The City College of The City University of New York, New York, New York 10031

linearized models. A good design procedure must take into account that there is a finite but unknown deviation between the model used for design and the real description of the process. This also applies to probabilistic models of the disturbance.

5. The Controller Must Be Reasonably Insensitive to Changes in System Parameters. It must be stable and perform well over a reasonable range of system parameters.

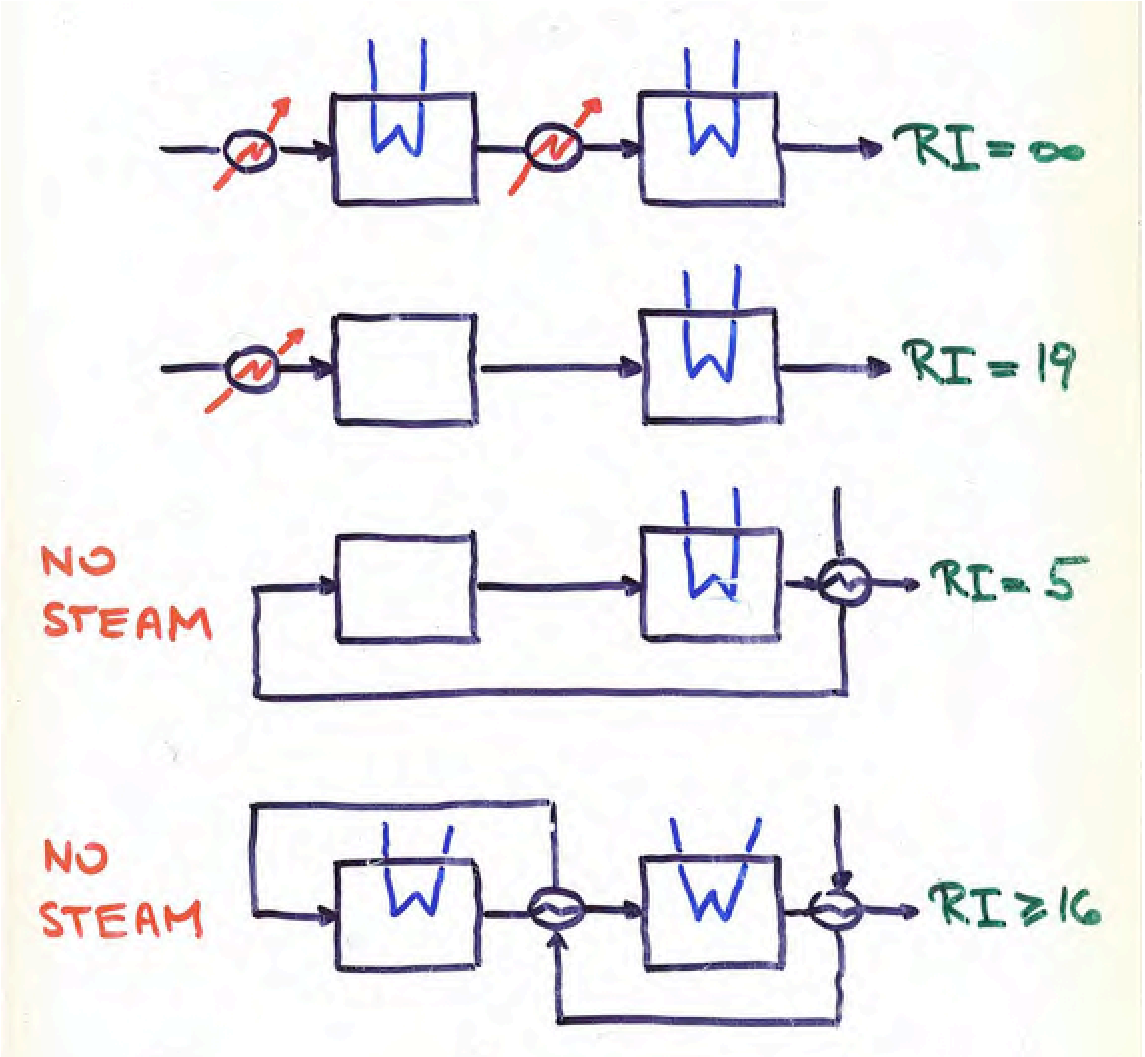
When we met…

- IFAC Workshop on Robust Control Systems, Interlaken, Switzerland, October 4-7, 1982. org. by J. Ackermann
- Participants: Barmish, Doyle, Frank, Kwakernaak, Looze, Mansour, Morari, Olbrot, Stein, Toedtli,…

Development and Application of Robust Control Techniques in Chemical Engineering

Manfred Morari
Chemical Engineering Department
University of Wisconsin – Madison

Controllability Assessment of Design Alternatives



Morari, 1981

ICI Ltd
Roger Benson
NERED, Teesside
Bodo Linnhoff
Corporate Lab, Runcorn

Controllability Assessment of Design Alternatives
Inherent Performance Limitations

Performance Limitations:

1) Time delays
2) RHP transmission zeros
3) Controller gain : $\| G_-^{-1} \| = \sigma_{min}^{-1}(G)$
4) Robustness : $\gamma(\omega) = \dfrac{\sigma_{max}(G)}{\sigma_{min}(G)}$ = cond. number

$$G = (I + \Delta)\tilde{G} \quad , \quad \|\Delta\| < \delta(\omega)$$

$$\| F G_-^{-1} \Delta G \| < 1 \qquad \text{nec. \& suff}$$

$$\Downarrow$$

$$\sigma_{max}(F) < \frac{1}{\gamma(\omega)} \cdot \frac{1}{\delta(\omega)} \qquad \text{suff. only}$$

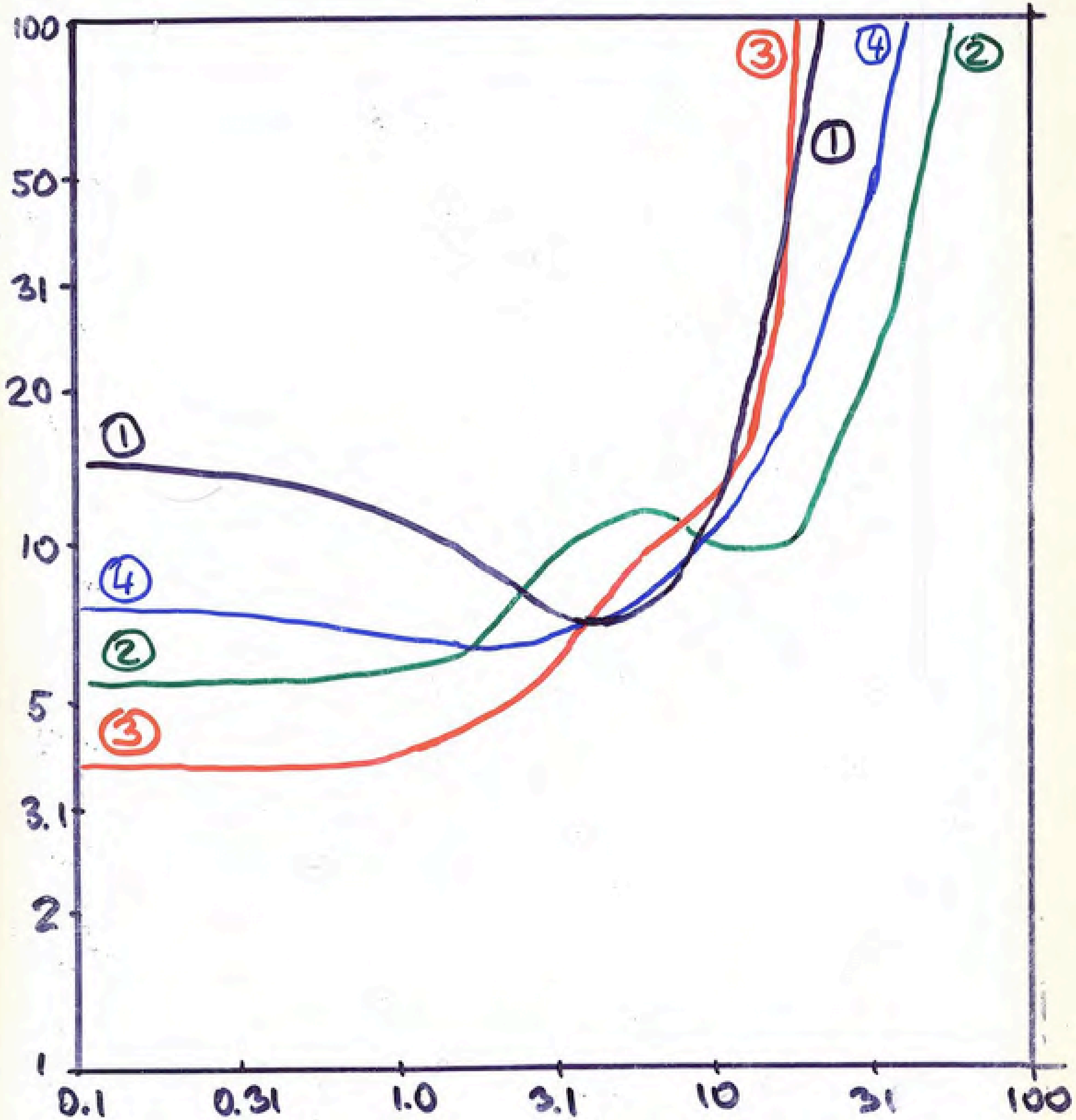
Robust efficiency and actuator saturation explain healthy heart rate control and variability

Na Li[a], Jerry Cruz[b], Chenghao Simon Chien[c,d], Somayeh Sojoudi[e], Benjamin Recht[f], David Stone[g], Marie Csete[h], Daniel Bahmiller[b], and John C. Doyle[b,c,i,1]

[a]School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138; [b]Department of Computing and Mathematical Science, California Institute of Technology, Pasadena, CA 91125; [c]Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125; [d]Advanced Algorithm Research Center, Philips Healthcare, Thousand Oaks, CA 91320; [e]Department of Neurology, New York University Comprehensive Epilepsy Center, New York University School of Medicine, New York, NY 10016; [f]Department of Electrical Engineering and Computer Sciences and Department of Statistics, University of California, Berkeley, CA 94720; [g]Departments of Anesthesiology and Neurosurgery and the Center for Wireless Health, University of Virginia School of Medicine, Charlottesville, VA 22908; [h]Huntington Medical Research Institutes, Pasadena, CA 91101; and [i]Department of BioEngineering, California Institute of Technology, Pasadena, CA 91125

$$\sigma_m = \frac{\lambda_{MAX}^{1/2}(\tilde{G}^*\tilde{G})}{\lambda_{MIN}^{1/2}(\tilde{G}^*\tilde{G})}$$

Different Designs



NSF Proposal 1985
Robust Controller Design for Systems with Constraints

NOTICE OF RESEARCH PROJECT
SCIENCE INFORMATION EXCHANGE
SMITHSONIAN INSTITUTION
NATIONAL SCIENCE FOUNDATION
PROJECT SUMMARY

SIE PROJECT NO
NSF AWARD NO

FOR NSF USE ONLY

DIRECTORATE/DIVISION	PROGRAM OR SECTION	PROPOSAL NO.	F.Y.

NAME OF INSTITUTION (INCLUDE BRANCH/CAMPUS AND SCHOOL OR DIVISION)

California Institute of Technology

ADDRESS (INCLUDE DEPARTMENT)

Chemical Engineering, 206-41
Pasadena, California 91125

PRINCIPAL INVESTIGATOR(S)

Manfred Morari, John C. Doyle, Carlos E. Garcia

TITLE OF PROJECT

Robust Controller Design for Systems with Constraints

TECHNICAL ABSTRACT (LIMIT TO 22 PICA OR 18 ELITE TYPEWRITTEN LINES)

For more than a decade practitioners have pointed at model uncertainty and operating constraints as the key difficulties in control design for chemical processes. The recently developed concept of a Structured Singular Value (SSV) holds great promise for accounting for model uncertainty in a systematic manner in control system design. The main reason for the popularity of the model predictive control techniques which have appeared in the last few years, is their ability to handle constraints. The objective of the work proposed here is to resolve a number of practical and theoretical issues regarding the SSV and model predictive control with particular emphasis on the control problems occuring in the chemical process industries. Theoretical advances, software development and applications to industrial plants are planned through a Industry-University Cooperative Research agreement with Shell Development Co.

NATIONAL SCIENCE FOUNDATION

PROPOSAL EVALUATION FORM

NSF Form 1B (4-80) Supersedes A...

PROPOSAL NO. CBT-8519896 | INSTITUTION California Inst of Tech | PLEASE ... 09/27

PRINCIPAL INVESTIGATOR Manfred Morari | NSF PROGRAM PROCESS & REACTION ENGINE

TITLE Industry - University Cooperative Research: Robust Controller Design for Systems with Constraints

COMMENTS (CONTINUE ON ADDITIONAL SHEET(S) AS NECESSARY)

QUALITY OF THE PROPOSED RESEARCH (INCLUDING BUDGET AND INSTITUTIONAL CAPABILITY):

This is the best control proposal I have seen in two decades. The PIs' not onl know what has been done, but what to do. At the same time, there is much room creative discovery while confronting realistic difficult problems. The industr involvement is solid enough to ensure quick reduction of theory to practice.

RECENT RESEARCH ACHIEVEMENTS OF THE PRINCIPAL INVESTIGATOR(S):

This is likely to enhance greatly the control engineer's ability to design for constrained, nonlinear systems -- The hard problems.

OVERALL RATING: [X] EXCELLENT [ ] VERY GOOD [ ] GOOD [ ] FAIR [ ] POOR

NATIONAL SCIENCE FOUNDATION

PROPOSAL EVALUATION FORM

NSF Form 1 (4/84) Supersedes All Previous Ed...

PROPOSAL NO. CBT-8519896 | INSTITUTION California Inst of Tech | PLEASE RETURN BY 09/27/85

PRINCIPAL INVESTIGATOR Manfred Morari | NSF PROGRAM PROCESS & REACTION ENGINE

TITLE Industry - University Cooperative Research: Robust Controller Design for Systems with Constraints

Please evaluate this proposal using the criteria presented on the back of this review form. Continue on additional sheet(s) as necessary.

The subject proposal is one of the best I have ever seen in the process control field. It should be funded as quickly as possible, at the highest possible level.

Impact of the proposed research on advancing the knowledge or technology base:

This proposed research, combining some of the best available academic tal... with skilled personnel from a major industrial corporation, is very likel... to have major impact on the process control field.

OVERALL RATING: [X] EXCELLENT [ ] VERY GOOD [ ] GOOD [ ] FAIR [ ] POOR

Impact of the base:

By its ver... will most... which will...

OVERALL RATING: [ ] EXCELLENT [X] VERY GOOD [ ]

OVERALL RATING: [ ] EXCELLENT [X] VERY GOOD [ ] GOOD [ ] FAIR [ ] POOR

OVERALL RATING: [X] EXCELLENT [ ] VERY GOOD [ ] GOOD [ ] FAIR [ ] POOR

OVERALL RATING: [X] EXCELLENT [ ] VERY GOOD [ ] GOOD [ ] FAIR [ ] POOR

OVERALL RATING: [ ] EXCELLENT [X] VERY GOOD [ ] GOOD [ ] FAIR [ ] POOR

NSF Proposal 1985
Robust Controller Design for Systems with Constraints

NATIONAL SCIENCE FOUNDATION

PROPOSAL EVALUATION FORM

NSF Form 1 (9/83)
Supersedes All Previous Editions

PROPOSAL NO.	INSTITUTION	PLEASE RETURN BY
CBT-8519896	California Institute of Tech	9/27/86

PRINCIPAL INVESTIGATOR	NSF PROGRAM
Manfred Morari	Process and Reaction Engineering

TITLE: Industry - University Cooperative Research: Robust Controller Design for Systems with Constraints

Please evaluate this proposal using the criteria presented on the back of this review form. Continue on additional sheet(s) as necessary.

SEE ATTACHED. 9 Pages

Impact of the proposed research on advancing the knowledge or technology base:

I don't believe it will have any useful impact.

OVERALL RATING: ☐ EXCELLENT ☐ VERY GOOD ☐ GOOD ☐ FAIR ☒ POOR

REVIEWER'S SIGNATURE | DATE | REVIEWER'S NAME (TYPED)

NSF Proposal 1985
Excerpts from the "Poor" Review

The proposal has many statements obviously made to impress the reader, but which do not stand up. For example on p. 2, "The

the values at s=0, which is ridiculous. These two claims of practicality are fallacious marketeering, unworthy even of ordinary **business** practice.

p. 11). Probably even here, the calculations are horrendous, or the results very poor. Again and again, they must admit that even their special (highly impractical) cases give conservative (meaning wasteful) results. I cannot at all see the usefulness of any of their work. So much complicated mathematics and so much hard work if numerics are tried, and for what-stability analysis which is conservative and which offers so little insight.

Caltech 1987

IFAC 1987, Munich

Uncertainty and…

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 33, NO. 12, DECEMBER 1988

Robust Control of Ill-Conditioned Plants: High-Purity Distillation

SIGURD SKOGESTAD, MANFRED MORARI, MEMBER, IEEE, AND JOHN C. DOYLE

S. Skogestad is with the Department of Chemical Engineering, Norwegian Institute of Technology, Trondheim, Norway.
M. Morari and J. C. Doyle are with the Department of Chemical Engineering, California Institute of Technology, Pasadena, CA 91125.

***Abstract*—Ill-conditioned plants are generally believed to be difficult to control. Using a high-purity distillation column as an example, the physical reason for the poor conditioning and its implications on control system design and performance are explained. It is shown that an acceptable performance/robustness trade-off cannot be obtained by simple loop-shaping techniques (via singular values) and that a good understanding of the model uncertainty is essential for robust control system design. Physically motivated uncertainty descriptions (actuator uncertainty) are translated into the H_∞/structured singular value framework, which is demonstrated to be a powerful tool to analyze and understand the complex phenomena.**

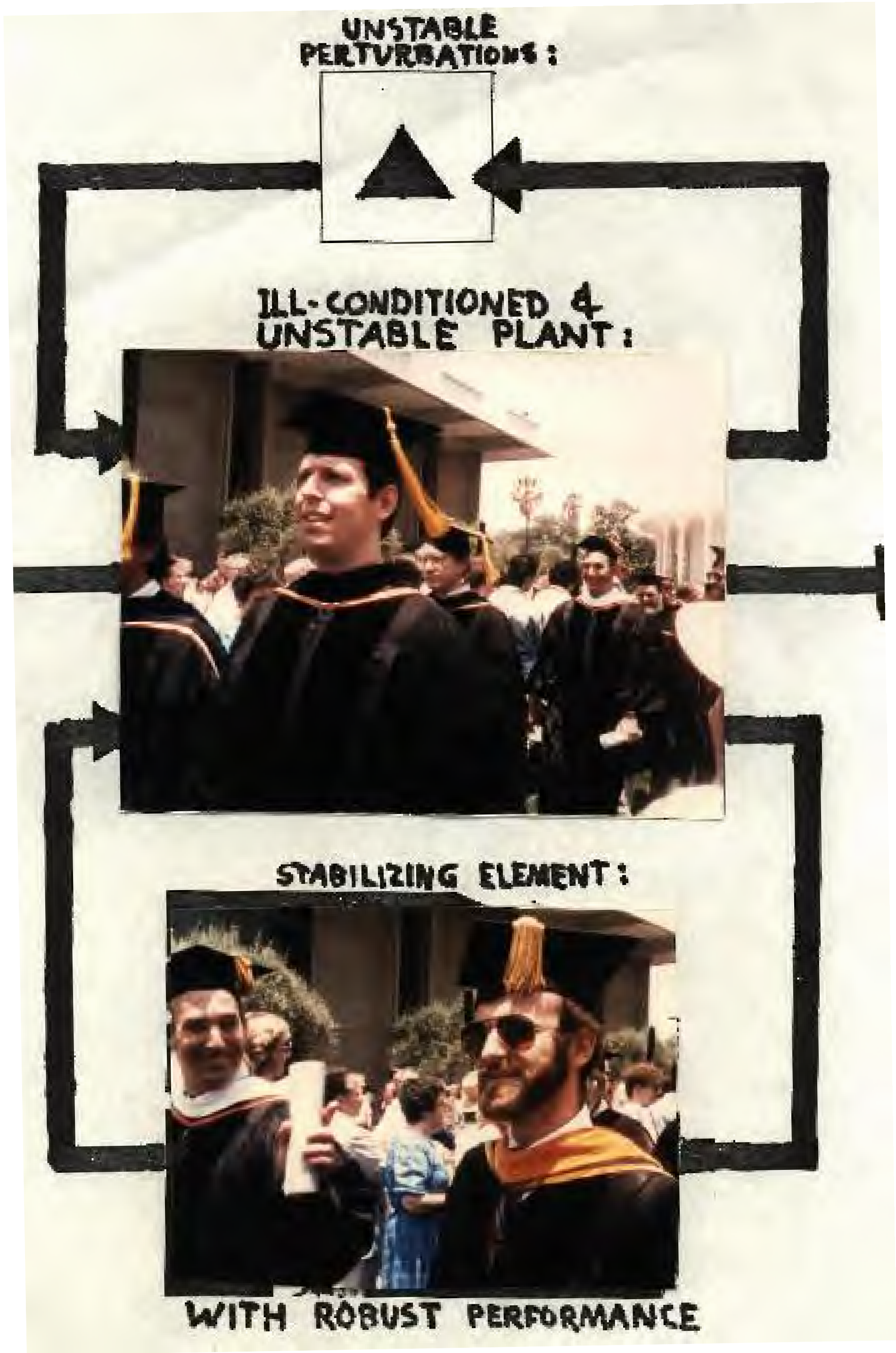
… Computation

Computational Complexity of μ Calculation

Richard P. Braatz, Peter M. Young,
John C. Doyle, and Manfred Morari

Abstract— The structured singular value μ measures the robustness of uncertain systems. Numerous researchers over the last decade have worked on developing efficient methods for computing μ. This paper considers the complexity of calculating μ with general mixed real/complex uncertainty in the framework of combinatorial complexity theory. In particular, it is proved that the μ recognition problem with either pure real or mixed real/complex uncertainty is NP-hard. This strongly suggests that it is futile to pursue exact methods for calculating μ of general systems with pure real or mixed uncertainty for other than small problems.

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 39, NO. 5, MAY 1994



Skogestad, 1986

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Synthesis of Optimal Control Laws

Infinite-Horizon Optimal Control

$$J^*(x) = \min_{u_i \in U} \sum_{i=0}^{\infty} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$x_i \in X$$

Dynamic Programming

$$J^\star(x) = \min_u \, l(x,u) + J^\star(f(x,u))$$
$$\text{s.t. } (f(x,u), u) \in X \times U$$

- Challenge is computation!

Synthesis of Optimal Control Laws

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- Challenge is computation!
- Closed-form solution for linear systems, no constraints only: LQR,…

Synthesis of Optimal Control Laws

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$$\text{s.t. } (f(x,u),u) \in X \times U$$

Explicit calculation of control law $u^\star(x)$ *offline*

Model Predictive Control

$$J^*(x_0) = \min_{u_i} \sum_{i=0}^{N} l(x_i, u_i) + V_f(x_N)$$
$$\text{s.t. } (x_i, u_i) \in X \times U, \; x_N \in X_f$$
$$x_{i+1} = f(x_i, u_i)$$

Online optimization problem defines control action $u_0^\star(x)$

Model Predictive Control : Properties

Theory is well-established

Mayne, Rawlings, Rao, Scokaert (2000), *Automatica*

"MPC: Stability & Optimality (Survey Paper). "

- **Recursive feasibility**: Input and state constraints are satisfied
- **Stability** of the closed-loop system
  - $J^*(x)$ is a convex Lyapunov function

- **MPC = Nonlinear control synthesis with stability guarantees by design !!!**

- **Assuming** the real-time optimization problem is solved to ε-optimality

Verifiable Control Synthesis

Offline	Online
Explicit MPC	1st Order–Fast Gradient
Approx. Explicit MPC	Interior Point Opt.

Verifiable Control Synthesis

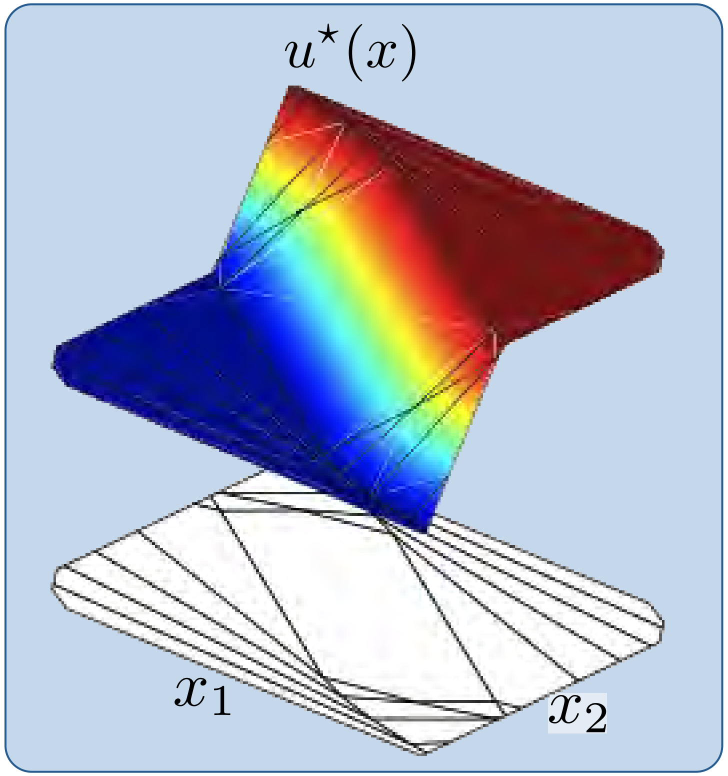
Offline	**Online**
Explicit MPC	1^{st} Order–Fast Gradient
Approx. Explicit MPC	Interior Point Opt.

Explicit MPC : Online => Offline Processing

- Optimization problem is parameterized by state
- Control law piecewise affine for linear systems / constraints
- Pre-compute control law as function of state x (parametric optimization)

Result : Online computation dramatically reduced

$$
\begin{aligned}
u^\star(x_0) = \operatorname*{argmin}_{u_i} \ & \sum_{i=0}^{N} l(x_i, u_i) + V_f(x_N) \\
\text{s.t. } & (x_i, u_i) \in X \times U \\
& x_{i+1} = f(x_i, u_i) \\
& x_N \in X_f
\end{aligned}
$$



[M.M. Seron, J.A. De Doná and G.C. Goodwin, 2000]
[T.A. Johansen, I. Peterson and O. Slupphaug, 2000]
[A. Bemporad, M. Morari, V. Dua and E.N. Pistokopoulos, 2000]

Verifiable Control Synthesis

Offline	Online
Explicit MPC	**1st Order Methods**
• < 5 states • Simple look-up • < μs sampling	
Approx. Explicit MPC	**Interior Point**

Verifiable Control Synthesis

Offline	Online
Explicit MPC • < 5 states • Simple look-up • < μs sampling	**1st Order Methods**
Approx. Explicit MPC • < 10 states • Specified complexity • < μs sampling	**Interior Point**

Verifiable Control Synthesis

Offline	Online
Explicit MPC	**1st Order Methods**
• < 5 states • Simple look-up • < µs sampling	• Any size • Simple and robust • µs – ms sampling
Approx. Explicit MPC	**Interior Point**
• < 10 states • Specified complexity • < µs sampling	• Any size • Highly accurate • ms sampling

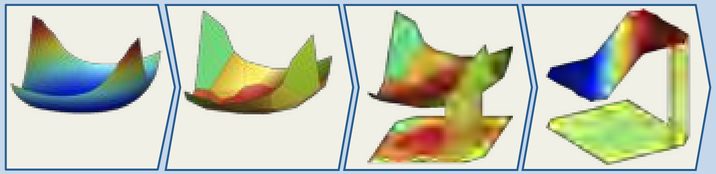
Computation / Software

Formal specification
- YALMIP
- HYSDEL
- Linear + Hybrid models

Control law
- Explicit MPC
- Fixed-complexity solutions

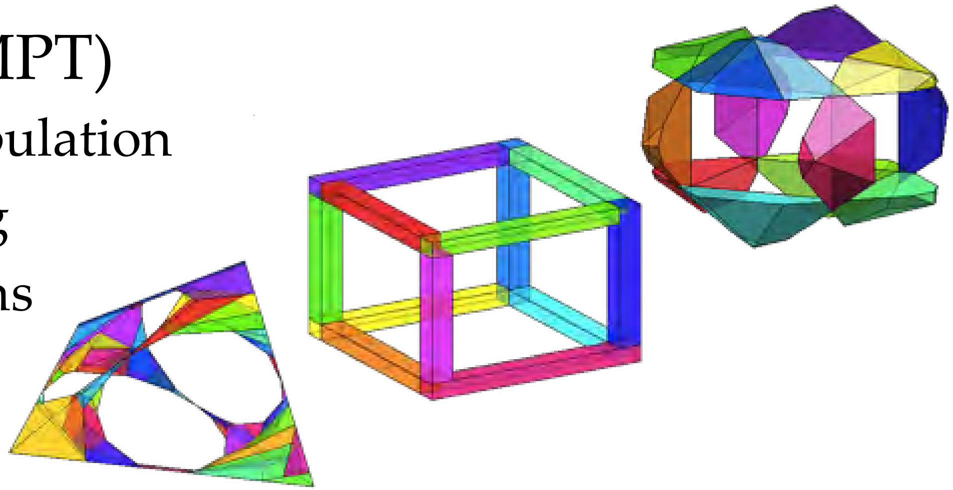
Software synthesis
- Real-time workshop
- Bounded-time solvers
- Verifiable code generation

Verified controller



Multi-Parametric Toolbox (MPT)

- (Non)-Convex Polytopic Manipulation
- Multi-Parametric Programming
- Control of PWA and LTI systems
- > 32,000 downloads to date



MPT 3.0 new in 2011

First Order Methods
FiOrdOs Code Generator

FiOrdOs
First Order Optimisation

- Matlab toolbox for automated C-code generation for first order methods
- Considered class of multi-parametric programs:

$$
\begin{array}{lll}
\min & \frac{1}{2}x^T H x + g^T x + c & H \succeq 0, H \neq 0 \\
\text{s.t.} & x \in \mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_N & \mathbb{X}_i\text{: elementary simple set, e.g.} \\
& Ax = b & \text{box, ball, simplex, LP-, SOCP-cone, …}
\end{array}
$$

Example: Code generation for *x*-axis MPC controller

$$\min_{U \in \mathbb{U}^N} \frac{1}{2} U^T \mathcal{T} U + (\mathcal{L}^T x)^T U$$

Matlab

```
% define simple set X
Box=EssBox(1,'l',umin,'u',umax);
UN=SimpleSet(N);
UN.addEss(1:N,Box);
% define problem, certify & code gen
OP=OptProb('X',UN,'H',' ','g','var');
Apor=AporAl(OP,'algo','fgm');
Apor.certify(ccsi on,'algo');
Apor.generateCode();
```

solver.c (1900 LOC)
solver.h (174 LOC)
mex.c (235 LOC)

[Master Thesis by F. Ullmann, 2011]

Interior Point Method
FORCES Code Generator

Multistage QCQP

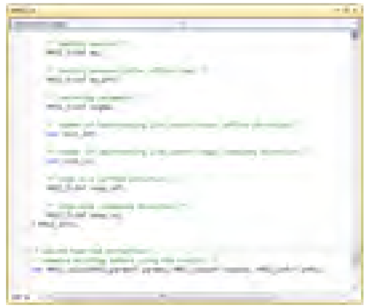
$$
\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^{N} \frac{1}{2} v_i^T H_i v_i + f_i^T v_i \\
\text{s.t.} \quad & \underline{z}_i \leq v_i \leq \bar{v}_i \\
& A_i v_i \leq b_i \\
& v_i^T Q_{i,j} v_i + l_{i,j}^T v_i \leq r_{i,j} \\
& C_i v_i + D_{i+1} v_{i+1} = c_i
\end{aligned}
$$

?

Embedded Hardware

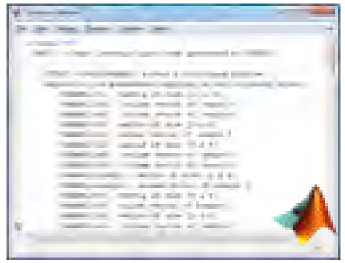
Generated Code

Solver (ANSI-C)



solver.h
solver.c
solver.m
solvermex.c
makemex

MATLAB MEX interface for rapid prototyping



Problem description

```
stage = MultiStageProblem(N+1);
for i = 1:N+1
% dimensions
stages(i).dims.n = 10;
stages(i).dims.r = 5;
stages(i).dims.lb = 3;

% cost
stages(i).cost.H = Hi;
stages(i).cost.f = fi;

% inequalities
stages(i).ineq.b.lbidx = 3:5;
stages(i).ineq.b.lb = zeros(3,1);

% equalities
stages(i).eq.C = Ci;
stages(i).eq.c = ci;
stages(i).eq.D = Di;
end
generateCode(stages);
```

FORCES

- C code generation of primal-dual Mehrotra interior point solvers
- LPs, QPs, QCQPs
- Parametric problems
- Multi-core platforms
- Library-free
- Available: forces.ethz.ch

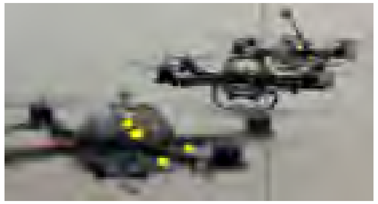
Some Early Users of FORCES

Nonlinear MPC & MHE with ACADO

Milan Vukov, KU Leuven, 2012

MPC for Wind Turbines

Marc Guadayol, ALSTOM, 2012



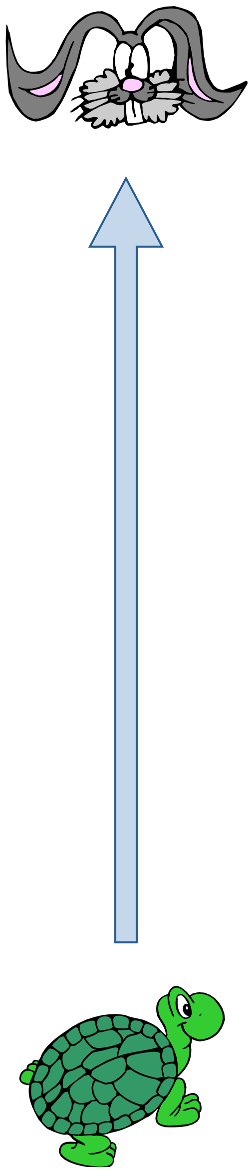
Quadrotor Control

Marc Müller, IDSC, ETH Zurich, 2012

Adaptive MPC for Belt Drives

Kim Listmann, ABB Ladenburg, 2012

Applications by the Automatic Control Lab



18 ns	Multi-core thermal management (EPFL) *[Zanini et al 2010]*
10 μs	Voltage source inverters *[Mariethoz et al 2008]*
20 μs	DC/DC converters (STM) *[Mariethoz et al 2008]*
25 μs	Direct torque control (ABB) *[Papafotiou 2007]*
50 μs	AC / DC converters *[Richter et al 2010]*
5 ms	Electronic throttle control (Ford) *[Vasak et al 2006]*
20 ms	Traction control (Ford) *[Borrelli et al 2001]*
40 ms	Micro-scale race cars
50 ms	Autonomous vehicle steering (Ford) *[Besselmann et al 2008]*
500 ms	Energy efficient building control (Siemens) *[Oldewurtel et al 2010]*

Embedded Conic Solver

[A. Domahidi, E. Chu, S. Boyd, ECC 2013]

- Goal: solve sparse second-order cone programs on embedded systems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & Gx \preceq_K h \end{array} \qquad \mathbf{K} \triangleq \mathbf{K}_1 \times \mathbf{K}_2 \times \ldots \mathbf{K}_N \text{ where } \mathbf{K}_i = \begin{cases} \mathbf{R}_+ \\ \mathbf{Q}^{n_i} \end{cases}$$

$$\text{and } \mathbf{Q}^{n_i} \triangleq \{(x_0, x_1) \in \mathbf{R} \times \mathbf{R}^{n_i - 1} \mid x_0 \geq \|x_1\|_2\}$$

- Applications: robust & soft-constrained MPC w/ guarantees, min. fuel descent, optimal power flow, robust beam forming, portfolio selection, machine learning (robust SVMs, group lasso) + all QPs, LPs, QCQPs

- Solver implementation (primal-dual IPM):
  - ~800 lines of ANSI C, detects infeasibility
  - Interfaces: MATLAB, Python, Java, .NET, Julia, Scala, CVX, CVXPY, Yalmip, Spark/MLlib, Breeze
  - Fastest free SOCP solver →
  - Widely used, e.g. by Verizon

github.com/ifa-ethz/ecos

1000 var. problem:

Mosek	0.04 s
Gurobi	0.08 s
SeDuMi	0.16 s
SDPT3	0.55 s
ECOS	**0.09 s**

ECOS

Maintained by embotech

Spinoff ETH zürich

Brightbox Technologies Inc.
MPC for Building Energy Mgt

- Flawless operation in several commercial bldgs.
- Most complex building: 8 packaged units and 600 vav boxes
  - 18,176 signals processed every 5 min.
  - MPC: >300,000 vars. and >500,000 constraints (sampling time 5 mins)

AUTOMATEDLOGIC®
CORPORATION

MPC: State of the Art

- MPC (on-line opt) advanced from process control brute force to theoretically founded method of choice in many application areas
- Synthesis of nonlinear controllers with guarantees
- Correct by design, not synthesis based on analysis.
- Computation technology is not limiting the application of (linear/linearized) MPC at any speed for any size problem

- When and where to employ MPC in industry is still a matter of judgment (modeling, maintenance, robustness)

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Some Open Research Areas in Control

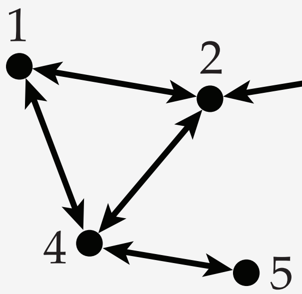
- Distributed systems with communication constraints
- Systems with discrete decisions and switched systems
- Systems with constraints and uncertainty
- Supervisory control systems

Some Open Research Areas in Control

- Distributed systems with communication constraints
- Systems with discrete decisions and switched systems
- Systems with constraints and uncertainty
- Supervisory control systems

Cooperative Distributed MPC

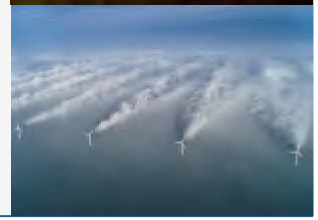
M dynamically coupled systems, locally constrained



$$x_i^+ = \sum_{j \in \mathcal{N}_i} A_{ij} x_j + B_i u_i$$

$$(x_i, u_i) \in \mathcal{X}_i \times \mathcal{U}_i$$

e.g. $\mathcal{N}_2 = \{1, 2, 3, 4\}$



e.g. power systems, irrigation systems, traffic networks, etc.

Communication Constraint

- Systems can communicate only if they are dynamically coupled
- No central coordination

Cooperative Distributed MPC

- "Distributed": Each system does local computations
- "Cooperative": Each system communicates with neighbors only to solve global optimization problem iteratively

Distributed Optimization Requires Structure

- Many distr. optimization methods available, see e.g. [Bertsekas et al., 1989]
- Methods allow for global optimization without central coordination
- Methods require structure in the global optimization problem

$$
\begin{aligned}
U^*(x) \in \arg\min_{U} \; & \boxed{V_\mathrm{f}(x(N))} + \sum_{k=0}^{N-1} l(x(k), u(k)) \\
\text{s.t. } & x(0) = x \;, \\
& x(k+1) = Ax(k) + Bu(k) \;, \\
& (x(k), u(k)) \in \mathcal{X} \times \mathcal{U} \;, \\
& \boxed{x(N) \in \mathcal{X}_\mathrm{f}} \;.
\end{aligned}
$$

Structured

Structured

Problem: Stability and feasibility enforcing components, i.e. terminal cost/set, are usually unstructured

→ MPC stability theory to be adapted to communication constraints

Feasibility and Stability for General MPC
Construction of Terminal Cost and Set

Closed-loop Stability under MPC

The closed-loop system is stable if $\forall x \in \mathcal{X}_\mathrm{f}$ it holds that

$$\kappa_\mathrm{f}(x) \in \mathcal{U} \quad ,$$

$$x \in \mathcal{X} \quad ,$$

Feasibility

$$Ax + B\kappa_\mathrm{f}(x) \in \mathcal{X}_\mathrm{f} \quad ,$$

$\mathcal{X}_\mathrm{f}$ pos. invariant

$$V_\mathrm{f}(Ax + B\kappa_\mathrm{f}(x)) - V_\mathrm{f}(x) \leq -l(x, \kappa_\mathrm{f}(x))$$

$V_\mathrm{f}(x)$ Lyapunov

Construction of terminal cost and set

1. *Quadratic terminal cost:*

$$\min_{P_\mathrm{f}, K_\mathrm{f}} \; -\log\det P_\mathrm{f}^{-1}$$

$$\text{s.t. } (A + BK_\mathrm{f})^T P_\mathrm{f} (A + BK_\mathrm{f}) - P_\mathrm{f} \leq -Q - K_\mathrm{f}^T R K_\mathrm{f}$$

LMI via Schur complement.

2. *Ellipsoidal terminal set:*

$$\max_{\alpha} \; \alpha$$

$$\text{s.t. } \mathcal{X}_\mathrm{f} = \{x \in \mathbb{R}^n \mid x^T P_\mathrm{f} x \leq \alpha\}, \;\; \mathcal{X}_\mathrm{f} \subseteq \mathcal{X}, \;\; K_\mathrm{f}\mathcal{X}_\mathrm{f} \subseteq \mathcal{U}; \;\; \mathcal{X}, \mathcal{U} \text{ polytopic}$$

LP via support function of $\mathcal{X}_\mathrm{f}$.

Feasibility and Stability for Distributed MPC
Structured Terminal Cost and Set

Suggestion

- Terminal cost: $V_\mathrm{f}(x) = \sum_{i=1}^{M} V_{\mathrm{f},i}(x_i)$, each $V_{\mathrm{f},i}(x_i)$ decreasing locally
- Terminal set: $\mathcal{X}_\mathrm{f} = \mathcal{X}_{\mathrm{f},1} \times \ldots \times \mathcal{X}_{\mathrm{f},M}$

Without using communication such a structure can often not be designed

Alternative

- Allow $V_{\mathrm{f},i}(x_i)$ to increase as long as $V_\mathrm{f}(x)$ decreases:

$$V_{\mathrm{f},i}(x_i^+) - V_{\mathrm{f},i}(x_i) \leq -l_i(x_{\mathcal{N}_i}, K_{\mathcal{N}_i} x_{\mathcal{N}_i}) + \gamma_i(x_{\mathcal{N}_i})$$

$$\sum_{i=1}^{M} \gamma_i(x_{\mathcal{N}_i}) \leq 0$$

- Allow time-varying terminal sets:

$$\mathcal{X}_{\mathrm{f},i}(\alpha_i) = \{x_i \in \mathbb{R}^{n_i} | V_{\mathrm{f},i}(x_i) \leq \alpha_i\}, \quad \alpha_i^+ = \alpha_i + \gamma_i(x_{\mathcal{N}_i})$$

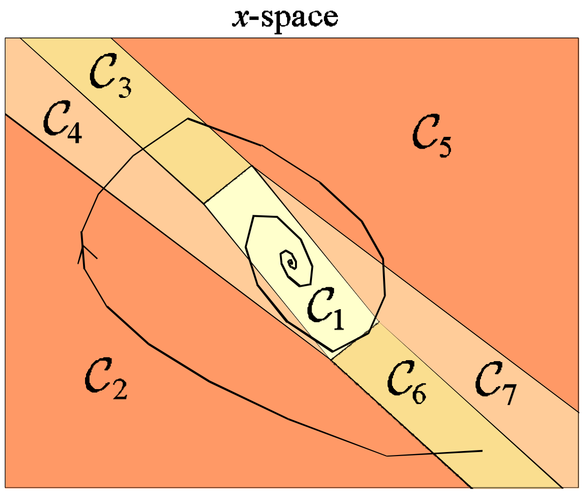
For linear systems, quadratic cost and polytopic constraints:
$V_{\mathrm{f},i}(x_i)$, $\gamma_i(x_{\mathcal{N}_i})$ *constructed via distributed LMI, resp. LP [Conte et al., 2012]*

Some Open Research Areas in Control

- Distributed systems with communication constraints
- Systems with discrete decisions and switched systems
- Systems with constraints and uncertainty
- Supervisory control systems

PWA Hybrid Models

- Piecewise affine (PWA) systems
- Polyhedral partition of state space
- Affine dynamics on reach region



$$\mathcal{C}_r \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^n \mid H_r \begin{bmatrix} x \\ u \end{bmatrix} \leq K_r \right\}$$

$$x(t+1) = A_r x(t) + B_r u(t) + f_r \quad \text{if} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{C}_r$$

MLD Hybrid Model

Discrete time linear dynamics and logic can be combined into **Mixed Logical Dynamical (MLD) form**

[Bemporad & Morari, 1999]

$$
\begin{aligned}
x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \\
y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \\
E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5
\end{aligned}
$$

$$
x, y, u = \begin{bmatrix} \star_c \\ \star_\ell \end{bmatrix}, \star_c \in \mathbb{R}^{n_c}, \star_\ell \in \{0,1\}^{n_\ell}, \;\; z \in \mathbb{R}^{r_c}, \;\; \delta \in \{0,1\}^{r_\ell}
$$

For MLD models all analysis and synthesis problems can be solved via Mixed Integer Linear/Quadratic Programs.

Speedup of software for MILP in 15 years

Linear Program	x 1000
Integer Program	x 100 – 1000
Computers	x 1000
Overall	**x 100 million**

Integer Programming

Preprocessing	x 2
Heuristics	x 1.5
Cutting Planes	x 50

Source: *Bixby, Gu, Rothberg, Wunderlich 2004*

Reminder
Constrained Optimal Control of Linear Systems

Constrained Optimal Control Problem

$$J(x) = \min_u \sum_{k=1}^{N} x_k^T Q' x_k + u_k^T R' u_k$$

$$x_{k+1} = A x_k + B u_k$$

$$x_k \in \mathcal{X}, \; u_k \in \mathcal{U}$$

Parametric QP

$$u(x_0) = \arg\min_u u^T Q u + x_0^T R u + s^T u$$

$$F u + G x_0 \leq h$$

$u(x_0)$ is a cont. polyhedral piecewise affine (PPWA) function of x_0

Idea: Model PPWA system as solution to optimal control problem

[Hempel, Goulart, Lygeros, IEEE-TAC, 2014]

Inverse Optimization System Models

Model PPWA system as solution to optimal control problem

Theorem:

Dynamics of any continuous PWA system can be expressed through parametric QP with current (x, u) as a parameter.

$$x^+ = A_i x + B_i u + f_i \qquad (x, u) \in \Omega_i$$

$$\Longleftrightarrow$$

$$x^+ \in T \, \underset{z}{arg\, min} \; J(z, (x, u)) \quad \text{s.t. } (z, (x, u)) \in \Gamma$$

- Use Karush-Kuhn-Tucker conditions to represent optimality
- Hybrid dynamics represented by complementarity conditions

Inverse Optimization System Models

Inverse parametric quadratic programming model:

$$
\begin{aligned}
x^+ &= Tz^* \\
z^* &\in \arg\min_{z} \ \frac{1}{2} z^T Q z + \begin{bmatrix} x^T & u^T \end{bmatrix} R z + s^T z \\
&\quad \text{s.t.} \ \ Fz + G \begin{bmatrix} x \\ u \end{bmatrix} \le h
\end{aligned}
$$

Equivalent complementary formulation:
(from KKT conditions)

$$
\begin{aligned}
x^+ &= Tz^* \\
Qz^* &+ R^T \begin{bmatrix} x \\ u \end{bmatrix} + s + F^T \lambda = 0 \\
0 &\le \lambda \ \perp \ h - Fz^* - G \begin{bmatrix} x \\ u \end{bmatrix} \ge 0
\end{aligned}
$$

Constrained Optimal Control Problem for PPWA Systems

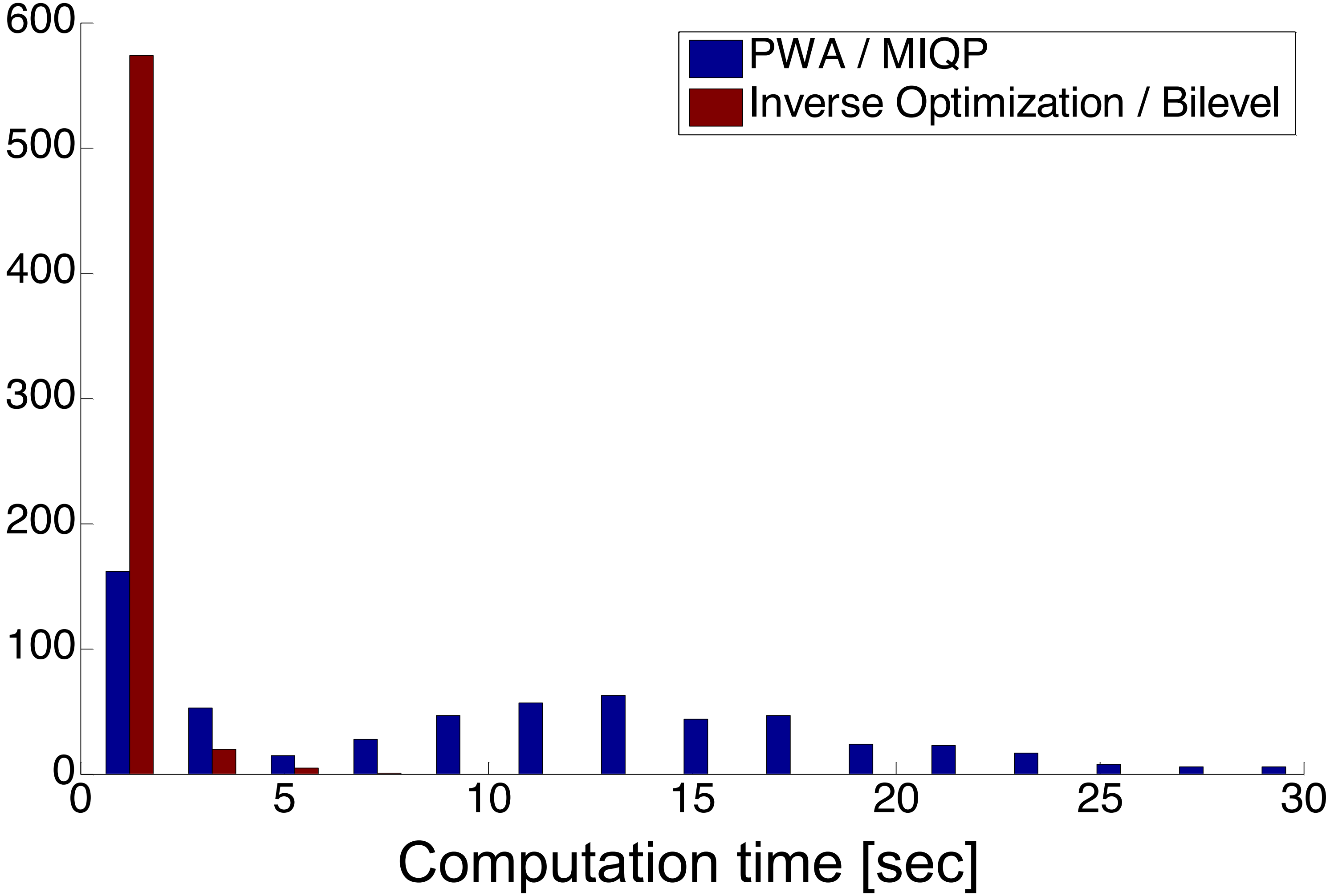
$$\min_{U} \quad x_N^T P x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$$
$$\text{s.t.} \quad (x_k, u_k) \in \mathcal{X}, \quad x_N \in \mathcal{T}$$

PWA model dynamics → MLD form
PWA model dynamics → Inverse optimization

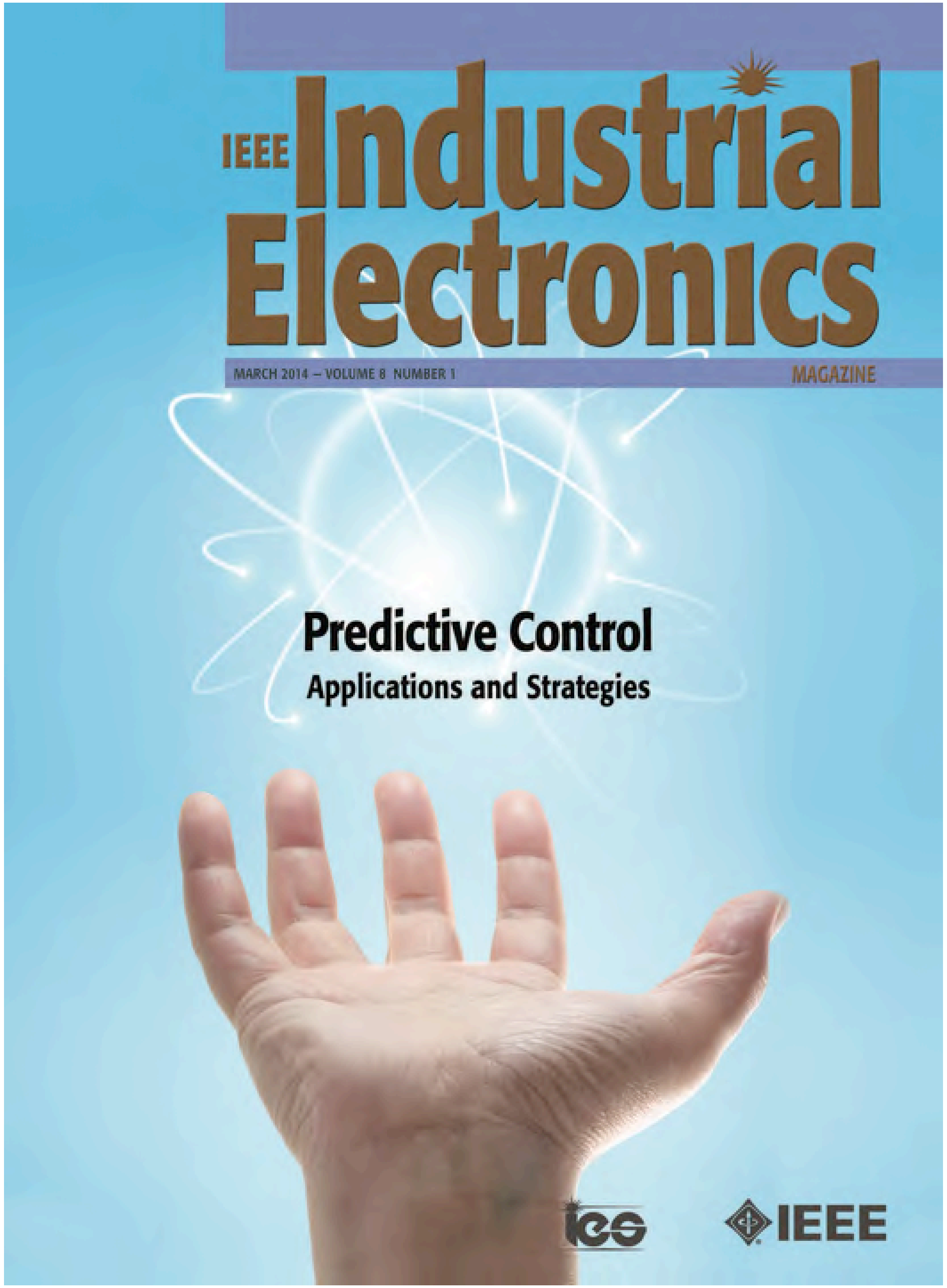
20 example systems:

- 6 states, 3 inputs, 7 regions
- Prediction horizon $N = 10$
- 30 different initial states x_0
- MLD solved with CPLEX (MIQP solver)
- Inverse optimization solved with IPOPT (NLP solver)

Computation Times for N = 10, 600 instances



Inverse optimization model solved faster than PWA model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Predictive Control
Applications and Strategies
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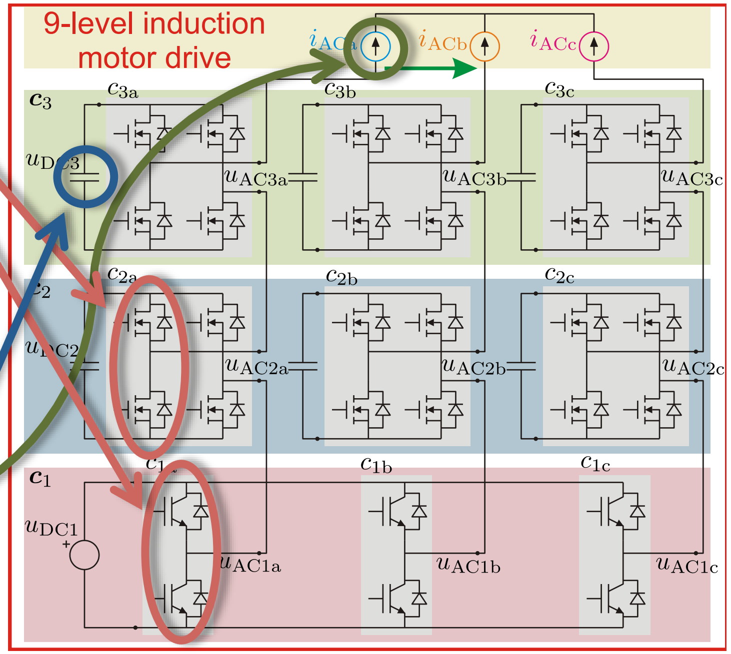
MIP in power electronics applications

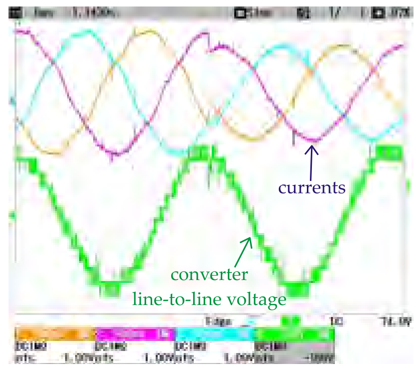
- New multilevel topologies emerging for high efficiency and power quality

15 independent pairs of switches operated at frequency > 1kHz, Horizon=50

Control:

- 6 capacitor voltages
- 3 motor currents





- Performance improvement requires accounting for binary nature of manipulated variables
- Need fast MIP solver to optimize performance in real-time

Some Open Research Areas in Control

- Distributed systems with communication constraints
- Systems with discrete decisions and switched systems
- Systems with constraints and uncertainty
- Supervisory control systems

Control of systems with uncertainty and constraints

Uncertain Constrained System

$$x^+ = (A + A_L \Delta A_R)x + (B + B_L \Delta B_R)u + Ew$$

$$x \in \mathcal{X}, \;\; u \in \mathcal{U}, \quad \forall (\delta, \Delta) \in \mathcal{W}$$

Controlling systems with *both* constraints *and* uncertainty is very difficult.

Uncertainty Models:

- Parametric / multiplicative uncertainties (Δ terms)
- Additive uncertainties (w terms)

Design Objectives:

- Robust or probabilistic constraint satisfaction
- Robust performance in some sense (H_2, H_∞ etc)

Solvable by dynamic programming <u>in principle</u>.

Models and available methods

Different ways to characterize the multiplicative or additive uncertain terms:

- Bounded uncertainty : uncertainties known only to live in a set
- Known distribution : uncertainty distribution can be perfectly modelled
- Partial moments: limited distributional information (mean and variance)

Limited successes to date:

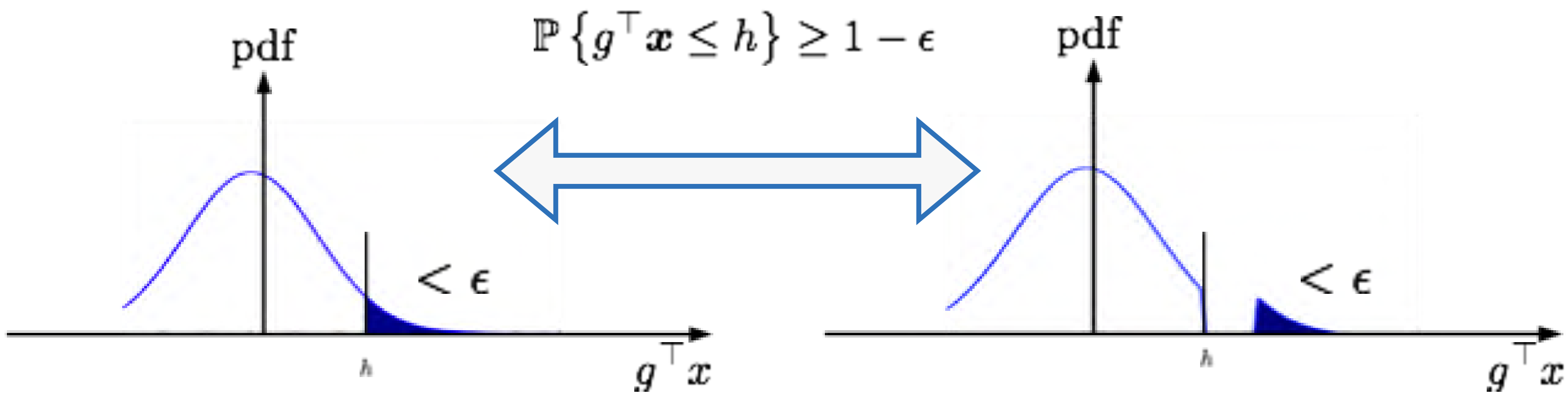
- Additive + bounds: Disturbance feedback / tube-based methods *[Goulart, Kerrigan, Maciejowski 2006; Mayne, Seron, Rakovic 2005]*
- Multiplicative + bounds : LMI-based methods *[Kothare, Balakrishnan, Morari, 1996; Cannon, Kouvaritakis 2005]*
- Multiplicative + known distribution : Scenario-based linear design methods *[Calfiore, Campi 2006; Calfiore, Fagiano 2013]*
- Additive + partial moments : Distributionally robust linear design methods *[Van Parys, Kuhn, Goulart, Morari 2014]*

Problems with chance constraints

A typical chance constraint condition:

$$\mathbb{P}\{\boldsymbol{x}_t \in \mathbb{X}\} \geq 1 - \epsilon$$

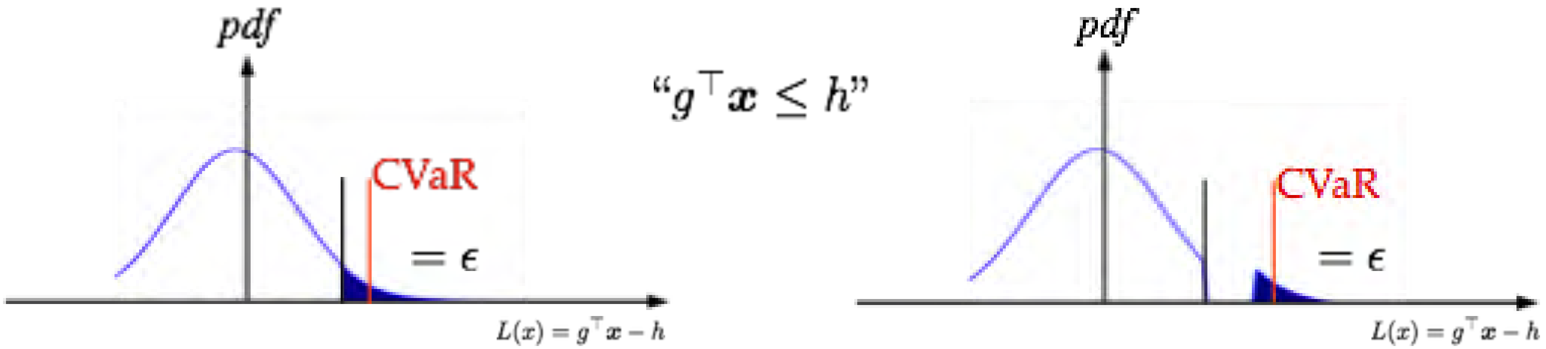
- No control over severity of constraint violation in outliers.



Most optimization-based approaches are based on sampling of uncertainty for finite horizon problems + MPC.

Conditional Violation at Risk constraints

CVaR is the center of mass of the ϵ-tail:



$\mathbb{X}$ is the zero sub-level set of the function L:

$$\boxed{x \in \mathbb{X} \iff L(x) \le 0}$$

- The function L quantifies constraint violation severity.

CVaR bounds imply chance constraint satisfaction:

$$\boxed{\mathbb{P}\text{–CVaR}_\epsilon\left(L(\boldsymbol{x})\right) \le 0 \implies \mathbb{P}\{\boldsymbol{x} \in \mathbb{X}\} \ge 1 - \epsilon}$$

Distributionally Robust Control

CVaR Constrained Design Problem

$$
\begin{aligned}
\inf_{\pi \in \Pi_\infty} \quad & \lim_{t \to \infty} \quad \mathbb{E}_{\mathbb{P}} \left\{ x_t^\top Q x_t + u_t^\top R u_t \right\} \\
\text{s.t.} \quad & x_{t+1} = A x_t + B u_t + E w_t, \\
& \limsup_{t \to \infty} \ \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\text{-CVaR}_\epsilon \left(L\left(x_t\right)\right) \le 0.
\end{aligned}
$$

Problem features:

- Optimal solution is a linear state feedback controller.
- The set $\mathcal{P}$ is the *distributional ambiguity set.*
- Robust CVaR constraint bounds the worst-case *severity of outliers.*
- Optimal solution can be computed by solving an SDP.
- Similar results for output feedback case.

[Van Parys, Kuhn, Goulart, Morari 2014]

Open questions for uncertain constrained systems

- Output feedback is mostly not well understood.
- Receding horizon methods for chance-constrained problems are mostly sampling based, with few infinite horizon results.
- Few clear connections to classical linear design methods.
- Many competing uncertainty models and numerical approaches, most of which are mutually incompatible.
- Most methods are sub-optimal : how can we measure their performance relative to the best possible controller?

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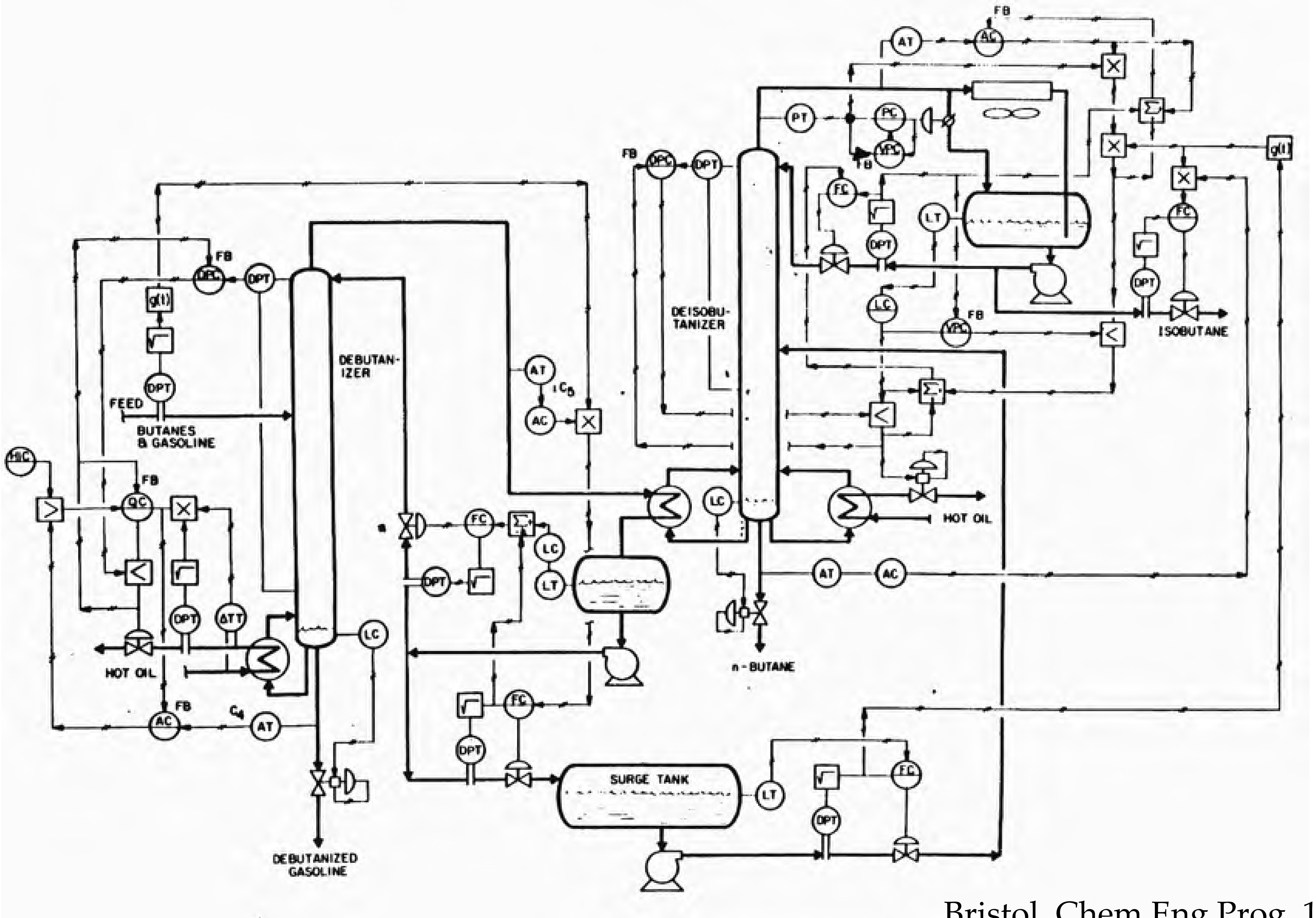
A typical Piping & Instrumentation Diagram



Bristol, Chem.Eng.Prog. 1980

Supervisory Control Logic

- Goals
  - Optimization: Adapt control targets for economic optimization
  - Constraint Management: obey operational constraints
  - Sequence transitions, e.g. start-up, shut down, reaction to failure,…

- Requirements
  - Robustness

- Problems
  - Analysis
  - Synthesis

Formal Verification of Embedded Software in Model Based Design

- Model checking of safety properties for Simulink Models
- Avionics distributed control system complexity:
  - 10K-250K simulink blocks
  - 40k-150K binary raw variables
  - Hundred to few thousand bin's after *simplification/abstraction*
- Automotive single controller complexity:
  - 5K-80K simulink blocks
  - Few thousand bin's after *simplification/abstraction*
- FormalSpecsVerifier tool environment (NuSMV)

Source: Alberto Ferrari

Advanced Laboratory on Embedded Systems S.r.l.

United Technologies

Conclusions

- Themes of Uncertainty and Computation
- For implementation MPC is alternative of choice, but open issues:
  - Communication constraints
  - Switches (incl supervisory control)
  - Uncertainty
- Match insight from “other methods” to MPC implementation