Sufficient Statistics for Decentralized Decision

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On the 20th anniversary of CDS at Caltech

Outline

- abstractly describe controller structure
- *separation* structure
- *sufficient statistics*
- decentralized generalization

State-space results

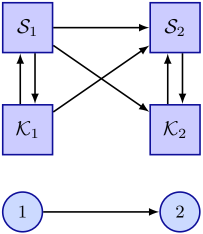
Example: two-player state-feedback control

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_{11} & 0\\ B_{21} & B_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + w$$

minimize

$$\lim_{T\to\infty} \frac{1}{T}\,\mathrm{E}\int_0^T \|Fx(t) + Du(t)\|^2\,dt$$

player 1 measures x_1
player 2 measures x_1, x_2



$$u_1 = K_{11}x_1 + K_{12}x_{2|1}$$
$$u_2 = K_{21}x_1 + K_{22}x_{2|1} + J\big(x_{2|1} - x_2\big)$$

Swigart and Lall, ACC 2010, Matni and Doyle, CDC 2013

Example: two-player output-feedback control

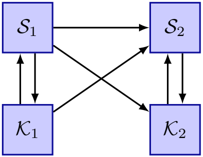
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_{11} & 0\\ B_{21} & B_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + w$$

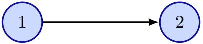
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}C_{11} & 0\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + v$$

minimize

$$\lim_{T\to\infty} \frac{1}{T}\,\mathrm{E}\int_0^T \|Fx(t) + Du(t)\|^2\,dt$$

player 1 measures y_1
player 2 measures y_1, y_2





$$u_1 = K_{11}x_{1|1} + K_{12}x_{2|1}$$

$$u_2 = K_{21}x_{1|1} + K_{22}x_{2|1} + J\left(x_{2|1} - x_{2|2}\right) + H\left(x_{1|1} - x_{1|2}\right)$$

Lessard and Lall, Allerton 2011

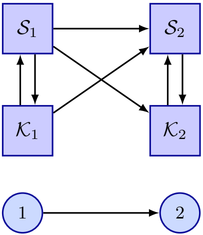
Example: nonlinear systems

$$x_1^{t+1} = f_1(x_1^t, u_1^t, w_1^t)$$
$$x_2^{t+1} = f_2(x_1^t, x_2^t, u_1^t, u_2^t, w_2^t)$$

minimize

$$\sum_{t=0}^{T} \mathrm{E}\, g(x^t, u^t)$$

player 1 measures x_1
player 2 measures x_1, x_2



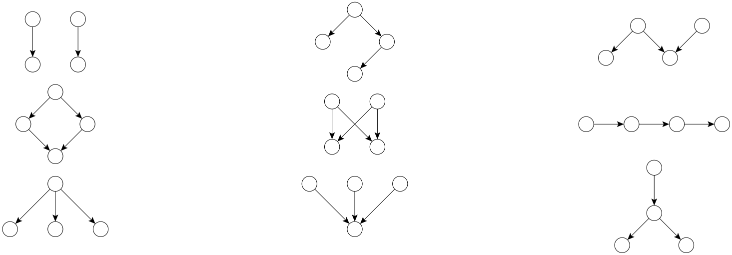
$$u_1^t = \mu_1^t(x_1^t, p_{2|1}^t)$$
$$u_2^t = \mu_2^t(x_1^t, x_2^t, p_{2|1}^t)$$

Wu and Lall, AAAI 2010

Example: transitively closed graphs

$$\dot{x} = Ax + Bu$$

A, B are in the incidence algebra



optimal u_i is a linear function of $x_{|\text{ancestors}} - x_{|(\text{ancestors} \cup \text{self})}$

Swigart thesis 2010, Shah and Parrilo 2010, Lamperski and Doyle 2011

Status

- some LQ problems: we know the structure
- others: we know the controller, but not the structure
- few nonlinear problems

Sufficient Statistics

Stochastic decision problems

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u = \mu(y) \end{array}$$

- given joint pdf of x, y, find best μ
- *e.g.*, estimation: with $c = \|x - u\|$
- can generate y, x with a model, *e.g.*, $y = Ax + w$
- hypothesis testing, classification, detection, decision, etc.,

Sufficient statistics

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u = \mu(y) \end{array}$$

$s = g(y)$ is called *sufficient* for $x \mid y$ if

$$y \perp\!\!\!\perp x \mid s$$

- equivalently $\mathrm{prob}(x \mid s, y)$ does not depend on y
- conditional distribution $x \mid y$ depends only on s
- Fisher 1922, Kolmogorov 1942

Sufficient statistics

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u = \mu(y) \end{array}$$

$s = g(y)$ is called *sufficient* for $x \mid y$ if

$$y \perp\!\!\!\perp x \mid s$$

optimal policy has the form $u = \mu(s)$

- does not depend on cost function
- s may be much smaller than y
- saves communications, storage, sensors, ...

Examples: sufficient statistics

- Gaussian noisy measurements $y_i = x + w_i$ then $s = \sum_i y_i$
- multiplicative uniform noise: $y_i = xw_i$ then $s = \max_i y_i$
- y_i is Bernoulli with $\mathrm{prob}(y_i = 1 \mid x) = x$, then $s = \sum_i y_i$
- if $y = Ax + w$ and w is Gaussian, then $s = A^T y$
- y_i has discrete uniform distribution on $[0, x]$, then $s = \max_i y_i$
  called *German tank problem*
- many others ...

Team Sufficiency

all results in Jeff Wu's thesis

Team decision problems

$$\begin{array}{ll} \text{minimize} & \mathrm{E}\, c(x, u_1, \ldots, u_n) \\ \text{subject to} & u_i = \mu_i(y_i) \end{array}$$

- given joint distribution $x, y_1, \ldots, y_n$, find n *policies* μ_i
- formulated by Marschak, 55
- quadratic and Gaussian: Radner, 62
- general case NP-hard, Tsitsiklis and Athans, 85
- H_2 model matching

Optimization

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{xyu} c_{xu} p_{xyu} \\ \text{subject to} & p_{xyu} = q_{xy} K_{yu} \\ & K_{yu} \text{ binary, stochastic} \end{array}$$

- easy; separate problem for each y
- LP relaxation corresponds to *randomized policies* $u = \mu(y, w)$

u is generated by $u = \mu(y, w)$ iff $u \perp\!\!\!\perp x \mid y$

Optimization

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{xyu} c_{xu} p_{xyu} \\ \text{subject to} & p_{xyu} = q_{xy} K^1_{y_1 u_1} \dots K^n_{y_n u_n} \\ & K^i_{y_i u_i} \text{ binary, stochastic} \end{array}$$

- relax the feasible set to convex hull

$$p_{xyu} \in \mathrm{co}\{ q_{xy} K^1_{y_1 u_1} \dots K^n_{y_n u_n} \mid K^i \text{ binary, stochastic} \} = H$$

- does not change cost

Team decisions

we say random variables $u_1, \ldots, u_n$ are a *team decision*, denoted

$$u_1, \ldots, u_n \perp\!\!\!\perp\!\!\!\perp x \mid y_1, \ldots, y_n$$

if $p_{xyu} \in \mathrm{co}\big\{q_{xy} K^1_{y_1 u_1} \ldots K^n_{y_n u_n} \mid K^i \text{ binary, stochastic}\big\}$

- $u_1, \ldots, u_n \perp\!\!\!\perp\!\!\!\perp x \mid y_1, \ldots, y_n$ if and only if u is generated by common randomness $u_i = \mu(y_i, w)$
- reduces to conditional independence in single player case
- $u_1, \ldots, u_n \perp\!\!\!\perp\!\!\!\perp x \mid y_1, \ldots, y_n$ implies $(u_1, \ldots, u_n) \perp\!\!\!\perp x \mid (y_1, \ldots, y_n)$

Relaxation

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u_1,\dots,u_n \perp\!\!\!\perp x \mid y_1,\dots,y_n \end{array}$$

- not an algorithm, but very useful definition

Multi-player sufficient statistics

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x, u_1, \ldots, u_n) \\ \text{subject to} & u_i = \mu_i(y_i) \end{array}$$

if $s_i = g_i(y_i)$, then $s_1, \ldots, s_n$ are called *team sufficient* for $x \mid y_1, \ldots, y_n$ if

$$y_1, \ldots, y_n \perp\!\!\!\perp x \mid s_1, \ldots, s_n$$

theorem: if s is team sufficient, then there exists an optimal policy of the form

$$\boxed{u_i = \mu_i(s_i)}$$

Multi-player sufficient statistics

- $s_1, \ldots, s_n$ is team sufficient if $y_1, \ldots, y_n \perp\!\!\!\perp x \mid s_1, \ldots, s_n$
- then there is a deterministic optimal controller $u_i = \mu_i(s_i)$
- *optimal* and *deterministic* even though s is defined in terms of
    - convex hull of feasible distributions
    - randomized policies

Example: Two players

suppose

$$s_1 \text{ is sufficient for } x, y_2 \mid y_1$$
$$s_2 \text{ is sufficient for } x, y_1 \mid y_2$$

then s_1, s_2 is team sufficient for $x \mid y_1, y_2$

for example, if x, y_1, y_2 are jointly Gaussian, then

$$s_1 = \mathrm{E}\left(\begin{bmatrix} x \\ y_2 \end{bmatrix} \;\middle|\; y_1\right) \qquad s_2 = \mathrm{E}\left(\begin{bmatrix} x \\ y_1 \end{bmatrix} \;\middle|\; y_2\right)$$

Example: Triangular

- measurements: $y_1 = z_1$ and $y_2 = (z_1, z_2)$
- suppose

$$r_1 \text{ is sufficient for } r_2 \mid z_1$$
$$r_2 \text{ is sufficient for } x \mid z_1, z_2$$

- then s_1, s_2 are team sufficient statistics for $x \mid y_1, y_2$, where

$$s_1 = r_1$$
$$s_2 = (r_1, r_2)$$

- Gaussian case

$$s_1 = \mathrm{E}(x \mid z_1)$$
$$s_2 = \big(\mathrm{E}(x \mid z_1), \mathrm{E}(x \mid z_1, z_2)\big)$$

Example: Quadratic cost

player 1 measures z_1 and player 2 measures z_1, z_2, cost is

$$\mathbf{E} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}^{\mathsf{T}} Q \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}$$

optimal policy is

$$\boxed{\begin{aligned} u_1 &= K_1 x_{|1} \\ u_2 &= K_2 x_{|1} + H(x_{|12} - x_{|1}) \end{aligned}}$$

where

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{bmatrix}^{-1} \begin{bmatrix} Q_{21} \\ Q_{31} \end{bmatrix} \qquad H = -Q_{33}^{-1} Q_{31}$$

Constructing sufficient statistics

for single player case, can construct from distribution:

- $\mathrm{prob}(y \mid x) = h(y) f\big(g(y), x\big)$ for some functions h, f
- the conditional distribution of $x \mid y$ is a function of s

many algebraic rules; write $\mathrm{suff}(x \mid y)$ for the set of sufficient statistics

- $h(s) \in \mathrm{suff}(x \mid y) \implies s \in \mathrm{suff}(x \mid y)$
- $s \in \mathrm{suff}(x \mid y) \implies s \in \mathrm{suff}\big(f(x, s) \mid y\big)$
- if $z \perp\!\!\!\perp y \mid x$ then $s \in \mathrm{suff}(x \mid y) \implies s \in \mathrm{suff}(x, z \mid y)$
- if $x \perp\!\!\!\perp u \mid y$ then $s \in \mathrm{suff}(x \mid y) \implies s \in \mathrm{suff}(x \mid y, u)$
- if $x \perp\!\!\!\perp z \mid y$ then $s \in \mathrm{suff}(x, z \mid y) \iff s \in \mathrm{suff}(x \mid y)$ and $s \in \mathrm{suff}(z \mid y)$

Constructing team sufficient statistics

elimination theorem: if

$$s_n \text{ is sufficient for } (x, y_1, \ldots, y_{n-1}) \mid y_n$$

$$(s_1, \ldots, s_{n-1}) \text{ is team sufficient for } (x, s_n) \mid y_1, \ldots, y_{n-1}$$

then

$$\boxed{s_1, \ldots, s_n \text{ is team sufficient for } x \mid y_1, \ldots, y_n}$$

- given a team sufficient statistic for $n-1$ players, constructs one for n players
- allows algebraic, inductive construction
- extensions of this result exist

Graphs example

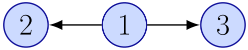
$$x_2 = f_{21}(x_1, w_2) \qquad z_1 = h_1(x_1, v_1) \qquad y_1 = z_1$$
$$x_3 = f_{31}(x_1, w_3) \qquad z_2 = h_2(x_2, v_2) \qquad y_2 = (z_1, z_2)$$
$$z_3 = h_3(x_3, v_3) \qquad y_3 = (z_1, z_3)$$

pick s_i so that

s_3 is sufficient for $x_1, x_3 \mid y_3$

s_2 is sufficient for $x_1, x_2 \mid y_2$

s_1 is sufficient for $s_2 \mid y_1$ and for $s_3 \mid y_1$



$s_1, (s_1, s_2), (s_1, s_3)$ are team sufficient for $x \mid y_1, y_2, y_3$

Dynamics

Dynamics

$$x^{t+1} = f(x^t, w^t)$$
$$y^t = h(x^t, w^t)$$

- well-known stochastic filter allows update of belief state $\mathrm{prob}(x^t \mid y^0, \ldots, y^t)$
- Kalman filter in linear Gaussian case
- sufficient statistics: if s^t is sufficient for $x^t \mid y^0, \ldots, y^t$ then

$\left(s^t, y^{t+1}\right)$ is sufficient for $x^{t+1} \mid y^0, \ldots, y^{t+1}$

Dynamics

Suppose we have dynamics with measurements

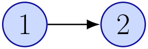
$$\begin{aligned} x^{t+1} &= f(x^t, w^t) \\ y_1^t &= h_1(x^t, w^t) \\ &\vdots \\ y_m^t &= h_m(x^t, w^t) \end{aligned}$$

if $s_1^t, \ldots, s_m^t$ is team sufficient for $x^t \mid y_1^{0:t}, \ldots, y_m^{0:t}$, then

$(s_1^t, y_1^{t+1}), \ldots, (s_m^t, y_m^{t+1})$ is team sufficient for $x^{t+1} \mid y_1^{0:t+1}, \ldots, y_m^{0:t+1}$

Example: Updating on graphs

$$x_1^{t+1} = f_1(x_1^t, w_1^t) \qquad z_1^t = h_1(x_1^t, v_1^t) \qquad y_1^t = z_1^t$$
$$x_2^{t+1} = f_2(x_1^t, x_2^t, , w_2^t) \qquad z_2^t = h_2(x_2^t, v_2^t) \qquad y_2 = (z_1^t, z_2^t)$$



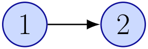
update team sufficient statistics

$$s_1^{t+1} \in \text{suff}(s_2^{t+1} \mid s_1^t, y_1^{t+1})$$
$$s_2^{t+1} \in \text{suff}(x^{t+1} \mid s_1^t, s_2^t, y_2^{t+1})$$

$s_1^t, (s_1^t, s_2^t)$ are team sufficient for $x^t \mid y_1^{0:t}, y_2^{0:t}$

Example: Updating on graphs

$$
\begin{aligned}
x_1^{t+1} &= f_1(x_1^t, w_1^t) & z_1^t &= h_1(x_1^t, v_1^t) & y_1^t &= z_1^t \\
x_2^{t+1} &= f_2(x_1^t, x_2^t, , w_2^t) & z_2^t &= h_2(x_2^t, v_2^t) & y_2 &= (z_1^t, z_2^t)
\end{aligned}
$$



in the Gaussian case

$$
\begin{aligned}
s_1^t &= \mathrm{E}(x^t \mid y_1^{0:t}) \\
s_2^t &= \mathrm{E}(x^t \mid y_2^{0:t})
\end{aligned}
$$

- player 1 estimates x given its information
- player 2 runs the same estimator as player 1, plus an additional one

Summary

- new concept: sufficient statistics for multi-player problems
  - reduction in size of states and storage
  - maintains optimality independent of cost
  - fundamental to state-space synthesis
  - see thesis by Jeff Wu, Stanford
- many algebraic tools
- constructive algorithms for certain graphs
- updating algorithms
- dynamic programming

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